Lecture Notes in Mathematics 1914

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Affine Density in Wavelet Analysis

Springer

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Until September 2007

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Library of Congress Control Number: 2007928330
Mathematics Subject Classification (2000): 42C40, 42C15, 94A12

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN 978-3-540-72916-7 Springer Berlin Heidelberg New York

DOI 10.1007/978-3-540-72949-5

Springer is a part of Springer Science+Business Media
springer.com

Typesetting by the author and SPi using a Springer LaTeX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 12070794 41/SPi 5 4 3 2 1 0

Dedicated to

my Parents

Preface

During the last 20 years, wavelet analysis has become a major research area in mathematics, not only because of the beauty of the mathematical theory of wavelet systems (sometimes also called affine systems), but also because of its significant impact on applications, especially in signal and image processing. After the extensive exploration of orthonormal bases of classical affine systems that has occupied much of the history of wavelet theory, recently both *wavelet frames* — redundant wavelet systems — and *irregular wavelet systems* — wavelet systems with an arbitrary sequence of time-scale indices — have come into focus as a main area of research. Two main reasons for this are to serve new applications which require robustness against noise and erasures, and to derive a deeper understanding of the theory of classical affine systems. However, a comprehensive theory to treat irregular wavelet frames does not exist so far. The main difficulty consists of the highly sensitive interplay between geometric properties of the sequence of time-scale indices and frame properties of the associated wavelet system.

In this research monograph, we introduce the new notion of affine density for sequences of time-scale indices to wavelet analysis as a highly effective tool for studying irregular wavelet frames. We present many results concerning the structure of weighted irregular wavelet systems with finitely many generators, adding considerably to our understanding of the relation between the geometry of the time-scale indices of these general wavelet systems and their frame properties.

This book is the author's Habilitationsschrift in mathematics at the Justus-Liebig-Universität Gießen. It is organized as follows. The introduction presents a detailed overview of the recent developments in the study of irregular wavelet frames and of the already quite established theory of the relation between Beurling density and the geometry of sequences of time-frequency indices of Gabor systems. Furthermore, it explains our main results in an informal way. Chapter 2 reviews the terminology and notations from frame theory as well as from wavelet and time-frequency analysis employed in this book.

The notion of weighted affine density, which will turn out to be a most effective tool for studying the geometry of sequences of time-scale indices associated with weighted irregular wavelet systems, will be introduced in Chapter 3. We illustrate the new notion by giving several examples. We further compare this notion of affine density with the affine density that was independently and simultaneously introduced by Sun and Zhou [119] and point out the advantages of our notion.

In Chapter 4, we prove that the notion of weighted affine density leads to very elegant necessary conditions for the existence of general wavelet frames on the sequence of time-scale indices. The usefulness of this notion is emphasized by its utility for the study of a rather technical-appearing hypothesis known as the *local integrability condition (LIC)* of a characterization result for weighted wavelet Parseval frames by Hernández, Labate, and Weiss [77]. In fact, we show that under a mild regularity assumption on the analyzing wavelets, the LIC is in fact solely a density condition.

Chapter 5 is devoted to the study of a quantitative relation between frame bounds and affine density conditions, since the complexity of frame algorithms is strongly related to the values of the frame bounds. A striking result here is a fundamental relationship between the affine density of the sequence of time-scale indices, the frame bounds, and the admissibility constant of a weighted irregular wavelet frame with finitely many generators. Several implications of this result are outlined, among which is the revelation of a reason for the non-existence of a Nyquist phenomenon for wavelet systems and the uniformity of sequences of time-scale indices associated with tight wavelet frames. In addition, we also present the first result in which the existence of particular wavelet frames is completely characterized by density conditions. The non-existence of very general co-affine frames is then shown to follow as a corollary.

In Chapter 6, we show that most irregular wavelet frames (and even wavelet Schauder bases) satisfy a so-called *Homogeneous Approximation Property (HAP)*. This property not only implies certain invariance properties under time-scale shifts when approximating with wavelet frames, but is also shown to have impact on density considerations. In addition to these main results, our techniques introduce some very useful new tools for the study of wavelet systems, e.g., certain Wiener amalgam spaces and — related with these objects — a particular class of analyzing wavelets.

Chapter 7 is devoted to the study of shift-invariance, i.e., invariance under integer translations, which is a desirable feature for many applications, since this ensures that similar structures in a signal are more easily detectable. The oversampling theorems from wavelet analysis show that most classical affine systems can be turned into a shift-invariant wavelet system with comparable frame properties. Most interestingly, the process also leaves density properties invariant, and the question concerning necessity of this fact for irregular wavelet systems arises. In this chapter we study the analog of this problem in time-frequency analysis and give a complete answer for irregular Gabor systems. Along the way we introduce a new notion of weighted Beurling density and derive extensions of results from H. Landau [97], and Balan, Casazza, Heil, and Z. Landau [7]. The results obtained in this chapter are not only interesting by itself, but can also be regarded as an important step towards the study of similar questions in wavelet analysis.

Acknowledgments

I am first and foremost indebted to Christopher Heil, who initiated the research for this book by inviting me as a Visiting Assistant Professor to the School of Mathematics at the Georgia Institute of Technology for the Fall Term of 2001. He has been a great friend and wonderful collaborator since then. I am deeply grateful to him for agreeing to include generalizations to multiple generators of our joint papers [73, 75], joint results on the HAP for Schauder bases (see Chapter 6), and a result from our joint paper [74] in this book. I also owe special thanks to Guido Weiss for inviting me to the Department of Mathematics at Washington University in St. Louis several times and for numerous inspiring discussions about my work. Further, I am indebted to Pete Casazza for introducing me to frame theory, for inviting me to the Department of Mathematics at the University of Missouri, and for being a great collaborator in several joint research projects.

I would like to thank Akram Aldroubi, Radu Balan, Marcin Bownik, Maria Carro, Ole Christensen, Charles Chui, Wojciech Czaja, Hans Feichtinger, Karlheinz Gröchenig, Jeffrey Hogan, Mihail Kolountzakis, Ilya Krishtal, Demetrio Labate, Zeph Landau, Ursula Molter, Amos Ron, Zouwei Shen, Javier Soria, Darrin Speegle, Yang Wang, and Edward Wilson for their interest in my work and for their many useful suggestions and hints. Further thanks go to Martin Buhmann and Christopher Heil for careful reading of parts of this book.

I am thankful to the Institutes of Mathematics at the Universität Paderborn and the Justus–Liebig–Universität Gießen for creating a pleasant working atmosphere. I would also like to acknowledge support from the German Research Foundation (DFG) in the form of DFG Research Fellowship KU 1446/5-1. This enabled me to spent one year in the U.S. during which a large portion of research for this book was developed. In this regard I am also thankful to the Department of Mathematics at Washington University in St. Louis and the School of Mathematics at the Georgia Institute of Technology for their hospitality and support during this year. I am indebted to Martin Buhmann and Tomas Sauer for supporting this research stay and consenting to my absence from the Institute of Mathematics at the Justus–Liebig–Universität Gießen during that period of time.

I would like to thank the staff at Springer-Verlag Heidelberg for editorial advise and help. I am grateful to Fernanda Laezza, Demetrio and Francesca

Labate, and Shelby, Bernie, and Lindsay Johnson for accepting me as a family member during my times in St. Louis and Atlanta and thereby contributing to make these stays a wonderful experience. Finally, I wish to thank my parents Hildegard and Norbert Kutyniok for their endless support which I can always count on.

Gießen, June 2006 *Gitta Kutyniok*

Contents

1

Introduction

1.1 Irregular Wavelet and Gabor Frames

Wavelet analysis has attracted rapidly increasing attention since Daubechies' groundbreaking book [41] in 1992 and is nowadays one of the major research areas in applied mathematics. The analyzing systems commonly used in wavelet analysis are the *classical affine systems*. Such a system consists of the collection of time-scale shifts of a function $\psi \in L^2(\mathbb{R})$, called the *analyzing wavelet*, associated with two parameters $a > 1$ and $b > 0$ and is given by

$$\{a^{-\frac{j}{2}}\psi(a^{-j}x - bk)\}_{j,k\in\mathbb{Z}}.$$

The origins of *time-frequency analysis* trace back to Gabor's article [59] on information theory, which appeared in 1946. This theory has also since become an important, independent branch of applied harmonic analysis. The function systems most often employed in this theory are the *regular Gabor systems*, which comprise the collection of time-frequency shifts determined by a function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$, specifically

$$\{e^{2\pi ibnx}g(x - ak)\}_{k,n\in\mathbb{Z}}.$$

There exist extensions to the higher dimensional situation for both systems, but in this introduction we restrict our discussion to the one-dimensional case for simplicity.

Both wavelet and Gabor systems play important roles in signal processing and data compression, e.g., in developing JPEG 2000, in solving MRI problems, and for the FBI fingerprint database (see, for instance, the books by Benedetto and Ferreira [8], Chui [25, 26], Feichtinger and Strohmer [56, 57], and Mallat [100]). These two types of systems are also a source of many intriguing mathematical problems and a useful tool in other areas of mathematics, see, e.g., the applications of wavelets to the study of Navier-Stokes or Euler equations (see, for instance, the books authored by Debnath [44] and Hogan and Lakey [82]).

Until some years ago the focus of research in wavelet analysis had been mainly on the construction of orthonormal bases. But recently the theory of frames, which generalize the notion of bases by allowing redundancy yet still providing a reconstruction formula, has been growing rapidly, since several new applications have been developed. Due to their robustness not only against noise but also against losses, and due to their freedom in design, frames — especially tight frames — have proven themselves an essential tool for a variety of applications such as, for example, nonlinear sparse approximation, coarse quantization, data transmission with erasures, and wireless communications (see, for instance, Benedetto, Powell, and Yilmaz [10], Candès and Donoho [13], Goyal, Kovačević, and Kelner [60], and Strohmer and Heath [116]). Gabor frames have already been studied for a longer time (cf. the books by Feichtinger and Strohmer [56, 57]), but recently also wavelet frames have become a main area of research in the wavelet community. (See, for example, the various papers authored by Chan, Chui, Czaja, Daubechies, Gröchenig, Han, He, Hernández, Labate, Maggioni, Riemenschneider, Ron, L. Shen, Z. Shen, Shi, Stöckler, Q. Sun, and Weiss [109, 67, 34, 27, 28, 29, 77, 32, 42, 19, 30, 31, 111].)

However, most results concerning wavelet and Gabor frames are restricted to the special cases of classical affine systems and regular Gabor systems. Recently, general irregular wavelet and Gabor systems, which can be built by using arbitrary time-scale or time-frequency shifts, have attracted increasing attention (see, for instance, the papers by Aldroubi, Balan, Cabrelli, Casazza, Christensen, Deng, Favier, Feichtinger, Felipe, Heil, Kaiblinger, Kutyniok, Lammers, Z. Landau, Molter, Ramanathan, Steger, W. Sun, and Zhou [107, 22, 23, 18, 118, 15, 73, 92, 119, 120, 1, 55, 121, 7, 93, 94, 95, 117]). An *irregular wavelet system* is determined by an analyzing wavelet $\psi \in L^2(\mathbb{R})$ and a sequence of time-scale indices $\Lambda \subseteq \mathbb{R}^+ \times \mathbb{R}$, regarded as a sequence in the affine group $\mathbb{A}$, and is defined by

$$\mathcal{W}(\psi, \Lambda) = \{a^{-\frac{1}{2}}\psi(a^{-1}x - b)\}_{(a,b)\in\Lambda}.$$

Given a function $g \in L^2(\mathbb{R})$ and a sequence of time-frequency indices $\Lambda \subseteq \mathbb{R}^2$, an *irregular Gabor system* is given by

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i b x} g(x - a)\}_{(a,b)\in\Lambda}.$$

The necessity of studying these general systems occurs since in practice a sequence of time-scale or time-frequency indices might be perturbed due to the impact of noise or other disturbances or may be directly imposed by the application at hand. Therefore results about the impact of properties of the sequence of time-scale or time-frequency indices on frame properties of the associated wavelet or Gabor system will turn out to be essentiell. Moreover, the study of irregular systems is very interesting from the mathematical point of view in deriving a deeper understanding of the theory of wavelet systems or

Gabor systems and, in particular, of the special case of classical affine systems or regular Gabor systems.

Later, it will become necessary to additionally equip the analyzing functions contained in the system with weights and also to consider systems with finitely many generators.

1.2 Density for Gabor Systems

Since time-scale and time-frequency indices associated with irregular wavelet and Gabor systems are initially completely arbitrary, we are led naturally to questions concerning the relation between their geometrical structure and the frame properties of the associated system. In order to put our results into perspective, let us review the density results that exist for the case of Gabor frames and the Heisenberg group.

Classical results are mostly concerned with regular Gabor systems, i.e., with rectangular lattices of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$. Baggett [4] and Daubechies [40] proved that if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a complete subset of $L^2(\mathbb{R})$ then necessarily $ab \leq 1$. Since every frame is complete (but not conversely), it follows as a corollary that if $ab > 1$, then $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ cannot form a frame. Baggett's proof uses deep results from the theory of von Neumann algebras, while Daubechies provided a constructive proof of this result by using signal-theoretic methods (the Zak transform). However, her result is restricted to the case that ab is rational. Daubechies also noted that a proof for general ab can be inferred from results of Rieffel [108] on C^*-algebras. Another proof of this result based on von Neumann algebras was given by Daubechies, H. Landau, and Z. Landau in [43], and a new proof appears in Bownik and Rzeszotnik [12].

H. Landau [98] extended the result on Gabor frames to much more general sequences Λ in $\mathbb{R}^2$, deriving a necessary condition for $\mathcal{G}(g, \Lambda)$ to be a frame in terms of the Beurling density of Λ, but requiring some restrictions on g and Λ. For the rectangular lattice case, Janssen [86] gave an elegant direct proof that if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a frame then $ab \leq 1$. This proof relies on the algebraic structure of the rectangular lattice $a\mathbb{Z} \times b\mathbb{Z}$ and the Wexler–Raz Theorem for Gabor frames. Perhaps the most elegant development along these lines was due to Ramanathan and Steger [107]. They proved that all Gabor frames $\mathcal{G}(g, \Lambda)$, without restrictions on $g \in L^2(\mathbb{R})$, but only for separated sequences $\Lambda \subseteq \mathbb{R}^2$, satisfy a certain *Homogeneous Approximation Property (HAP)*. This is a fundamental result that is of independent interest, and in particular they deduced necessary density conditions on the sequence of time-frequency indices of irregular Gabor frames as a corollary. Ramanathan and Steger were also able to recover the completeness result of Rieffel by using this technique as a main tool. However, the proof by Ramanathan and Steger required Λ to be uniformly separated. Christensen, Deng, and Heil removed this hypothesis in [22]. Also, [22] extended the result to higher dimensions and to finitely many generators, and made several other contributions. Christensen, Deng, and Heil derived the following general result on the density of Gabor systems which we state for simplicity only in the one-dimensional, singly generated case.

Theorem 1.1. *Let $g \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$ be given. Then the Gabor system $\mathcal{G}(g, \Lambda)$ has the following properties.*

(i) *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R})$, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$.*
(ii) *If $\mathcal{G}(g, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R})$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.*

Here $D^{\pm}(\Lambda)$ are the upper and lower Beurling densities of Λ, which measure in some sense the largest and smallest number of points of Λ that lie on average in unit squares. The group structure employed for this notion of density is the one coming from the Heisenberg group modulo its center. The cutoff density 1 is called the *Nyquist density.* In the special case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, we have $D^{\pm}(a\mathbb{Z} \times b\mathbb{Z}) = \frac{1}{ab}$. Gröchenig and Razafinjatovo adapted the Ramanathan/Steger argument to prove an analogous result for windowed exponentials in [66].

Ramanathan and Steger conjectured in [107] that Theorem 1.1(i) should be improvable to say that if $\mathcal{D}^-(\Lambda) < 1$ then $\mathcal{G}(g, \Lambda)$ is incomplete in $L^2(\mathbb{R})$. However, Benedetto, Heil, and Walnut [9] showed that the Rieffel result does not extend to non-lattices: there exist complete (but non-frame) Gabor systems with upper Beurling density ε. The counterexample built fundamentally on the work of H. Landau on the completeness of exponentials in $L^2(S)$ where S is a finite union of intervals. Another counterexample, in which Λ is a subset of a lattice, appears in Y. Wang [123]. Moreover, Olevskii and Ulanovskii [103, 104] constructed a system consisting solely of translates which is complete in $L^2(\mathbb{R})$, but even the upper density of the set of indices regarded as a subset of $\mathbb{R}^2$ equals zero.

Recently, Balan, Casazza, Heil, and Z. Landau showed that such necessary density conditions, including the Nyquist density cutoff, apply to a much broader class of abstract frames called *localized frames* [6]. Localized frames were also independently introduced by Gröchenig [64] for quite different purposes.

Finally, we remark that density theorems for Gabor frames $\mathcal{G}(g, \Lambda)$ generated by Gaussian functions g are related to density questions in the Bargmann–Fock spaces, see, e.g., Seip [112]. We further mention that the notion of Beurling density was also employed by Heil and Kutyniok [74] to derive conditions on the existence of frames and Schauder bases of windowed exponentials, and an adapted notion of density and a new notion of dimension were the main tools to study wave packet frames and also Gabor pseudoframes for affine subspaces in the papers by Czaja, Kutyniok, and Speegle [36, 37].

For more details and extended references we refer to the recent survey paper on the history of the density theorem for Gabor systems by Heil [72].

1.3 Geometry of Time-Scale Indices

It is natural to ask whether wavelet systems share similar properties, and the immediate answer is that there is clearly no exact analogue of the Nyquist density for wavelet systems. In particular, consider the case of the classical affine systems $\mathcal{W}(\psi, \Lambda)$ with dilation parameter $a > 1$ and translation parameter $b > 0$, i.e.,

$$\Lambda = \{(a^j, bk)\}_{j,k\in\mathbb{Z}}.$$

It can be shown that for *each* $a > 1$ and $b > 0$ there exists a wavelet $\psi \in L^2(\mathbb{R})$ such that $\mathcal{W}(\psi, \Lambda)$ is a frame or even an orthonormal basis for $L^2(\mathbb{R})$. In fact, the wavelet set construction of Dai, Larson, and Speegle [39] shows that this is true even in higher dimensions: wavelet orthonormal bases in the classical affine form exist for any expansive dilation matrix. For additional demonstrations of the impossibility of a Nyquist density, even given constraints on the norm or on the admissibility condition of the wavelet, see the example of Daubechies in [40, Thm. 2.10] and the more extensive analysis by Balan in [5].

However, the more general question remains: for what sequences $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ and what weights $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \dots, L$ is it possible to construct wavelet frames of the form

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell) = \bigcup_{\ell=1}^{L} \{w_\ell(a,b)^{\frac{1}{2}} a^{-\frac{1}{2}} \psi_\ell(a^{-1}x - b)\}_{(a,b)\in\Lambda_\ell}$$

with finitely many generators $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$? Two important examples of wavelet systems other than classical affine systems are the quasi-affine and co-affine systems.

Quasi-affine systems, introduced by Ron and Shen [109], are obtained by replacing the sequence Λ associated with a classical affine system by the new sequence

$$\Lambda = \{(a^j, bk)\}_{j<0,k\in\mathbb{Z}} \cup \{(a^j, a^{-j}bk)\}_{j\geq 0,k\in\mathbb{Z}},$$

and using the weight function

$$\begin{aligned} w(a^j, bk) &= 1, && j < 0,\ k \in \mathbb{Z}, \\ w(a^j, a^{-j}bk) &= a^{-j}, && j \geq 0,\ k \in \mathbb{Z}. \end{aligned}$$

In other words, "extra" elements are added to an affine system, and additionally the norms of the extra elements are adjusted. Ron and Shen proved that if a is an integer and $b = 1$ then an affine system is a frame if and only if the quasi-affine system is a frame. The utility of the quasi-affine system is that it is *shift-invariant*, i.e., integer translation-invariant, unlike the original classical affine system. Shift-invariance, i.e., invariance under integer translations, is a desirable feature for many applications, since it ensures that similar structures in a signal are more easily detectable. Quasi-affine systems were also studied in the papers by Bownik [11], Chui, Shi, and Stöckler [35], Gressman, Labate, Weiss, and Wilson [61], and Johnson [88].

Co-affine systems were studied recently by Gressman, Labate, Weiss, and Wilson [61]. If we write an affine system as $\{D_{a^j} T_k \psi\}_{j,k\in\mathbb{Z}}$, where D_{a^j} and T_k are the appropriate dilation and translation operators, then the associated co-affine system is $\{T_k D_{a^j} \psi\}_{j,k\in\mathbb{Z}}$. This amounts, in the terminology of this book, to taking

$$\Lambda = \{(a^j, a^{-j}k)\}_{j,k\in\mathbb{Z}},$$

and $w = 1$. It was shown in [61] that such a system $\mathcal{W}(\psi, \Lambda, w)$ can *never* form a frame for $L^2(\mathbb{R})$, and, moreover, this impossibility remains even when allowing weights of the form $w(a^j, a^{-j}k) = w(a^j)$. An extension of this result to higher dimensions was derived by Johnson [90].

Considering the Gabor situation we come to the conclusion that a notion of density for wavelet systems, despite the lack of a Nyquist density, should be exactly the right method to explain, for instance, the difference between affine/quasi-affine and co-affine systems, but even more to relate frame properties of a wavelet system with properties of the associated sequence of time-scale indices. It was already conjectured by Daubechies in [41, Sec. 4.1], that the value $\frac{1}{b \ln a}$ might play the role of a density for classical affine systems, since it is an ubiquitous constant in a variety of formulas in wavelet analysis. For example, if $\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k\in\mathbb{Z}})$ is a tight frame for $L^2(\mathbb{R})$ and $\int_0^\infty |\hat{\psi}(\xi)|^2/|\xi| d\xi = 1$, then the frame bounds are exactly $\frac{1}{b \ln a}$. In [113], Seip introduced a notion of density for Bergman-type spaces on the unit disk, and it is possible to derive some density results for wavelet frames $\mathcal{W}(\psi, \Lambda)$ generated by certain wavelets ψ from those results. Some preliminary results relating density to wavelet frames also appeared in a paper by Olson and Seip [105], but until now there has been no general theory of density properties of wavelet frames; there did not even exist a notion of density for general irregular wavelet systems.

To build such a theory is a fascinating challenge, since the situation for wavelet systems is much more delicate than the one for Gabor systems due to the non-commutativity of the affine group. Results in this direction should lead to a much deeper understanding of the geometrical structure of the time-scale indices associated with a wavelet frame, thereby also delivering tools to construct irregular wavelet frames and to examine their stability.

Summarizing, this research monograph has the following aims.

- ▷ Derive a notion of weighted affine density for weighted sequences of time-scale indices of weighted irregular wavelet frames with finitely many generators in such a way that classical affine systems possess a uniform density equal to the ubiquitous constant $\frac{1}{b \ln a}$.
- ▷ Study whether the non-existence of co-affine frames is related to density properties.
- ▷ Derive necessary and sufficient density conditions for the existence of weighted irregular wavelet frames with finitely many generators.

- ▷ Relate the density of the weighted sequences of time-scale indices to the frame bounds of weighted irregular wavelet frames with finitely many generators.
- ▷ Reveal reasons why a Nyquist phenomenon does not exist for wavelet systems.
- ▷ Study the HAP for wavelet systems and its relation (or lack thereof) to density conditions.
- ▷ Study the affine density of a classical affine system and the weighted affine density of its associated quasi-affine system and examine whether their relation is enforced by the property that one system is a frame if and only if the other system is a frame.

1.4 Overview of Main Results

In the following we outline the organization of this book and present some of the highlights in an informal way.

In Chapter 2 we present some background and notation from frame theory and wavelet and time-frequency analysis which will be employed throughout. We further give a brief overview of Wiener amalgam spaces in the general setting of locally compact groups, since we will consider different group settings in this book. These spaces will serve as regularity conditions for analyzing wavelets as well as for Gabor generators.

We proceed in Chapter 3 to introduce the new notion of upper and lower weighted affine density for weighted irregular wavelet systems with finitely many generators and to study several of its properties. We show that it satisfies the property that classical affine systems possess a uniform affine density, i.e., upper and lower density coincide, and this density is exactly equal to the magical constant $\frac{1}{b \ln a}$. Moreover, we compare this density with another notion of density for wavelet systems simultaneously introduced by Sun and Zhou [119]. We show that for their density a weighted form has to be used to derive the same uniform density for the classical wavelet systems, thereby emphasizing our notion as more naturally in this sense.

In Chapter 4, we derive necessary conditions on the upper and lower weighted affine density for the existence of a weighted irregular wavelet frame with finitely many generators. These results only rely on conditions concerning finite upper and positive lower density, in this sense on *qualitative* density conditions. More precisely, we prove that if such a wavelet system possesses an upper frame bound, then necessarily the upper density has to be finite (Theorem 4.1). This result confirms the intuitive view of the density as the amount to which the time-scale indices are concentrated. We further show that provided that the wavelet system possesses a lower frame bound, then, under some hypotheses on the time-scale indices and with weights being equal to one, the lower density must be positive (Theorem 4.2). Another result in the same spirit only under a mild regularity hypothesis on the analyzing wavelets will be derived in Chapter 6 (Corollary 6.12). For our proofs we rely on techniques from frame theory, Fourier analysis, and complex analysis. We apply these results to oversampled affine systems, co-affine systems, and systems consisting only of dilations, obtaining some new results relating density to the frame properties of these systems.

Using the methods developed in the first part of this chapter we then study a certain rather technical-appearing hypothesis known as the *local integrability condition (LIC)* of a characterization result of weighted wavelet Parseval frames with finitely many generators and with arbitrary sequences of scale indices by Hernández, Labate, and Weiss [77], and show that under some mild regularity assumption on the analyzing wavelets, the LIC is solely a very natural density condition on the sequences of scale indices. More precisely, it will be shown that the LIC is in fact equivalent to the condition that the weighted sequences of scale indices possess a finite upper weighted density (Theorem 4.22). Using this new interpretation of the LIC, we further discuss when the characterization result holds.

Chapter 5 is devoted to the study of *quantitative* density conditions in which the precise value of the density plays a role. The systems under consideration are weighted wavelet systems with finitely many generators and with arbitrary weighted sequences of time and scale indices satisfying some mild condition on the weighted sequences of time indices. A striking result here is a fundamental relationship between the affine weighted density, the frame bounds, and the admissibility constants for the analyzing wavelets (Theorem 5.6).

Several applications of this result are discussed. We show that such an irregular wavelet frame can only be tight if its weighted sequence of time-scale indices has a *uniform* weighted affine density (Corollary 5.9). This result provides new insight into tight irregular wavelet frames, since the classical affine systems, no matter whether they are tight or not, always possess a uniform density. In particular, we show that the sequence of time-scale indices of an orthonormal wavelet basis of the considered form has a uniform affine density. However, we show that this density does not rely on the norm of the generator as in the Gabor case, but instead on the admissibility constant, thereby revealing one reason why there does not exist a Nyquist phenomenon for wavelet systems (Section 5.4). We further derive the first result in which the existence of particular wavelet frames is completely characterized by density conditions (Theorem 5.11). This result is especially surprising since density conditions are independent of the analyzing wavelet itself and do not capture local features of the time-scale indices; hence they appear almost too weak to serve as a sufficient condition. Finally, we use the fundamental relationship to prove that certain weighted irregular co-affine systems can never form a frame (Theorem 5.18).

In Chapter 6 we derive several results related to the *Homogeneous Approximation Property (HAP)* for irregular wavelet systems with finitely many generators. The (Weak and Strong) HAP for Gabor systems states that approximating with a particular finite set of Gabor elements is invariant under time-frequency shifts. Moreover, it is a key property which leads to necessary conditions for Gabor frames in terms of the Beurling density of the associated sequence of time-frequency indices of the generator. We will show that, with a mild regularity assumption on the analyzing wavelets, irregular wavelet frames with finitely many generators satisfy an analog of the Strong HAP with respect to the affine group (Theorem 6.10). We further prove that irregular wavelet frames with finitely many generators satisfying the Weak HAP must fulfill certain density conditions with respect to other wavelet Riesz bases (Theorem 6.11). As an application we obtain necessary conditions for the existence of such frames in terms of the affine density (Corollary 6.12), which should be compared to those derived in Chapter 4.

In the second part of Chapter 6 we consider irregular wavelet Schauder bases with finitely many generators, and in fact this is the first time that those systems are systematically studied. We obtain results in the same spirit as in the first part of this chapter on the density of these Schauder bases, where the Strong HAP has to be replaced by the Weak HAP (Theorems 6.15 and 6.16, and Corollary 6.17).

In addition to these main results, our techniques introduce some new tools for the study of wavelet systems. In particular, we show that any generator of a wavelet frame which satisfies our regularity assumption has a continuous wavelet transform that lies in a particular Wiener amalgam space (Theorem 6.4). We further show that the class of analyzing wavelets which satisfy this regularity condition is in particular dense in the set of admissible wavelets.

In Chapter 7 we deviate a little from our main line of investigation and devote this chapter to Gabor systems. We introduce and study a new notion of *weighted* Beurling density for multiple weighted sequences of time-frequency indices suited to the study of weighted irregular Gabor systems with finitely many generators. We further prove that using arbitrary piecewise continuous, positive functions of a particular amalgam space instead of just characteristic functions of different boxes to measure the density of a weighted sequence leads to exactly the same notion of Beurling density. We think this result (Theorem 7.15) is interesting in its own right, since it, in particular, implies that for functions contained in the modulation space M^1 the associated short-time Fourier transform can serve as a measuring function for (weighted) Beurling density.

Then we establish a useful relationship between this density, the frame bounds, and the norms of the generators for the Gabor frames under consideration (Theorem 7.19).

Shift-invariance, i.e., invariance under integer translations, is a desirable feature for many applications, since this ensures that similar structures in a signal are more easily detectable. Regular Gabor systems have the great advantage over wavelet systems in that they are almost always automatically shift-invariant. However, most irregular Gabor systems lack this desirable property. We discuss the terminology of shift-invariance for these systems and introduce a machinery to associate a shift-invariant Gabor system to an irregular Gabor system. We further show that, provided frame properties are to remain unchanged, the shift-invariant counterpart has to be equipped with weights and that density conditions must be imposed on the sequences of time-frequency indices of both systems (Theorem 7.28).

We complete this section by conjecturing that for wavelet systems results similar to the ones obtained in Chapter 7 on the density properties of Gabor systems and its shift-invariant counterparts also hold, most likely in some weaker form. Therefore one future project will be to study these questions in the wavelet setting. We also mention that in this book we consider the univariate situation for wavelet systems, where we derive a complete theory. In the Gabor case the treatment of the multivariate case does not differ much from the one of the univariate case. However, in wavelet analysis the situation is completely different. Recent studies by Speegle [115] show that even for some multivariate classical affine systems it is not clear whether there exists an analyzing wavelet so that the resulting wavelet system forms an orthonormal basis. To derive an appropriate notion of density for multivariate wavelet systems will become one further topic for future research.

2

Wavelet and Gabor Frames

In this section we give a brief survey of the main notations, definitions, and results from frame theory, wavelet analysis, and time-frequency analysis which will be used throughout the book. We conclude this chapter with a section on amalgam spaces in the setting of locally compact groups, since in the sequel amalgam spaces will be employed in different group settings.

2.1 Frame Theory

In this section we briefly recall the definition and basic properties of frames and Schauder bases in Hilbert spaces. For more information on frame theory we refer to the various books and papers authored by Casazza [14], Christensen [20, 21], Daubechies [41], Heil and Walnut [76], and Young [128], and concerning Schauder bases theory we refer to Heil [70], Lindenstrauss and Tzafriri [99], Marti [101], Singer [114], and Young [128].

Let $\mathcal{H}$ be a separable Hilbert space, and let I be an indexing set. A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\,\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\,\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.1}$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. If A and B can be chosen such that $A = B$, then $\{f_i\}_{i\in I}$ is a *tight frame*. If we can take $A = B = 1$, it is called a *Parseval frame*.

Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Then the *frame operator*

$$Sf = \sum_{i\in I} \langle f, f_i\rangle\, f_i$$

is a bounded, positive, and invertible mapping of $\mathcal{H}$ onto itself, which satisfies

$$A\,\mathrm{Id} \le S \le B\,\mathrm{Id},$$

where Id denotes the identity operator. The *canonical dual frame* is $\{\tilde{f}_i\}_{i\in I}$, where $\tilde{f}_i = S^{-1} f_i$. For each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{i\in I} \langle f, f_i\rangle \, \tilde{f}_i = \sum_{i\in I} \langle f, \tilde{f}_i\rangle f_i.$$

In the special case that $\{f_i\}_{i\in I}$ forms a Parseval frame, the frame operator S is the identity, the dual frame coincides with the frame itself, and the frame expansions reduce to $f = \sum_{i\in I} \langle f, f_i\rangle \, f_i$.

A sequence which satisfies the upper frame bound estimate in (2.1), but not necessarily the lower estimate, is called a *Bessel sequence* and B is a *Bessel bound*. In this case,

$$\Big\|\sum_{i\in I} c_i f_i\Big\|^2 \leq B \sum_{i\in I} |c_i|^2 \qquad \text{for any } (c_i)_{i\in I} \in \ell^2(I). \tag{2.2}$$

In particular, $\|f_i\|^2 \leq B$ for every $i \in I$.

A sequence $\{f_i\}_{i\in\mathbb{N}}$ is a *Schauder basis* for $\mathcal{H}$ if for each $f \in \mathcal{H}$ there exist unique scalars $c_i(f)$, $i \in \mathbb{N}$, such that

$$f = \sum_{i=1}^{\infty} c_i(f) f_i. \tag{2.3}$$

Then there exists a unique biorthogonal system $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, which is also a Schauder basis, called the *dual basis*, and which satisfies

$$f = \sum_{i=1}^{\infty} \langle f, f_i\rangle \; \tilde{f}_i = \sum_{i=1}^{\infty} \langle f, \tilde{f}_i\rangle \, f_i \quad \text{for all } f \in \mathcal{H}.$$

The associated *partial sum operators* are $S_N(f) = \sum_{i=1}^{N} \langle f, \tilde{f}_i\rangle \, f_i$ for $f \in \mathcal{H}$. The *basis constant* is the finite number $C = \sup_N \|S_N\|$. If for each $f \in \mathcal{H}$ the series $f = \sum_i c_i(f) \, f_i$ converges with respect to any ordering of the indices, then $\{f_i\}_{i\in\mathbb{N}}$ is called an *unconditional basis*. Consequently, for a Schauder basis the ordering in (2.3) can be crucial. If $0 < \inf_i \|f_i\| \leq \sup_i \|f_i\| < \infty$ then $\{f_i\}_{i\in\mathbb{N}}$ is a *bounded basis*. A sequence $\{f_i\}_{i\in\mathbb{N}}$ which is a Schauder basis for its closed linear span within $\mathcal{H}$, denoted by $\overline{\text{span}}_{i\in\mathbb{N}}\{f_i\}$, is called a *Schauder basic sequence*.

We conclude this section with the following well-known result concerning the relationship between Schauder bases, Riesz bases, and frames (compare Casazza [14] or Christensen [21]).

Proposition 2.1. *The following three statements are equivalent:*

(i) $\{f_i\}_{i\in\mathbb{N}}$ *is a Schauder basis and a frame for* $\mathcal{H}$*,*
(ii) $\{f_i\}_{i\in\mathbb{N}}$ *is a Riesz basis for* $\mathcal{H}$*,*
(iii) $\{f_i\}_{i\in\mathbb{N}}$ *is a bounded unconditional basis for* $\mathcal{H}$*.*

2.2 Wavelet Analysis

In this section we will focus on the basic definitions, notations, and results in wavelet analysis which will be used in the sequel. For more information on wavelet theory we refer the reader to the books by Chui [24], Daubechies [41], and Hernández and Weiss [79], and the papers authored by Heil and Walnut [76] and Weiss and Wilson [124]. Most of the following definitions can be generalized to higher dimensions, but since in this book we focus on the one-dimensional situation, we just state the one-dimensional definitions. Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the *affine group*, endowed with the multiplication

$$(a,b) \cdot (x,y) = \big(ax, \tfrac{b}{x} + y\big).$$

The identity element of $\mathbb{A}$ is $e = (1,0)$, and inverses are given by

$$(a,b)^{-1} = \big(\tfrac{1}{a}, -ab\big).$$

The left-invariant Haar measure on $\mathbb{A}$ is $\mu_{\mathbb{A}} = \frac{dx}{x}\,dy$. We denote the norm and inner product on $L^2(\mathbb{A})$ with respect to this Haar measure by $\|\cdot\|_{L^2(\mathbb{A})}$ and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{A})}$, respectively, whereas the norm and inner product on $L^2(\mathbb{R})$ will be denoted by $\|\cdot\|$ or $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$.

Let σ be the unitary representation of $\mathbb{A}$ on $L^2(\mathbb{R})$ defined by

$$(\sigma(a,b)\psi)(x) = \tfrac{1}{\sqrt{a}}\psi(\tfrac{x}{a} - b) = D_a T_b \psi(x),$$

where D_a denotes the *dilation operator* $D_a f(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a})$ and T_b denotes the *translation operator* $T_b f(x) = f(x-b)$.

For $f \in L^1(\mathbb{R}^d)$, we will use the following convention for the Fourier transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \langle x, \xi\rangle}\,dx.$$

Its extension to a unitary mapping from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ will also be denoted by $\hat{f}$. The inverse Fourier transform shall be denoted by $f^\vee$.

Given $\psi \in L^2(\mathbb{R})$, called an *analyzing wavelet*, the *continuous wavelet transform (CWT)* $W_\psi f$ of $f \in L^2(\mathbb{R})$ with respect to ψ is

$$W_\psi f(a,b) = \big\langle f, \sigma(a,b)\psi\big\rangle = \big\langle f, D_a T_b \psi\big\rangle = \tfrac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x}{a} - b)}\, dx.$$

We have

$$|W_\psi f(a,b)| \leq \|f\|_2 \, \|\psi\|_2 \quad \text{for all } (a,b) \in \mathbb{A}$$

and $W_\psi f \in C(\mathbb{A})$. However, W_ψ does not map $L^2(\mathbb{R})$ into $L^2(\mathbb{A})$ for each $\psi \in L^2(\mathbb{R})$. We say that $\psi \in L^2(\mathbb{R})$ is *admissible* if the *admissibility constant* C_ψ defined by

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi$$

is finite. This condition is also sometimes called the *admissibility condition*. In particular, this is equivalent to the condition that both integrals

$$C_\psi^- = \int_{-\infty}^{0} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi \quad \text{and} \quad C_\psi^+ = \int_{0}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi$$

are finite. We further set

$$L_A^2(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) : \psi \text{ is admissible}\}.$$

Note that if $\psi \in L^1(\mathbb{R}) \cap L_A^2(\mathbb{R})$, then we must have $\hat{\psi}(0) = 0$, since $\hat{\psi}$ is continuous. If ψ is admissible, then W_ψ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{A})$, cf. Heil and Walnut [76, Cor. 3.3.6]. Precisely, we have that if $\psi \in L_A^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, then

$$\|W_\psi f\|_{L^2(\mathbb{A})}^2 = C_\psi^+ \int_0^\infty |\hat{f}(\xi)|^2\, d\xi + C_\psi^- \int_{-\infty}^0 |\hat{f}(\xi)|^2\, d\xi \le C_\psi\, \|f\|_2^2 .$$

Furthermore, the roles of f and ψ can be interchanged by using the relation $W_f\psi(a,b) = \overline{W_\psi f((a,b)^{-1})}$. We remark that this lack of symmetry is due to the fact that the Haar measure on $\mathbb{A}$ is not unimodular.

The next lemma lists several useful equivalent forms of the CWT:

Lemma 2.2. *If $f, \psi \in L^2(\mathbb{R})$, then*

$$\begin{aligned} W_\psi f(a,b) &= \tfrac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x-ab}{a})}\, dx \\ &= \sqrt{a} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \overline{\hat{\psi}(a\xi)}\, e^{2\pi i a b \xi}\, d\xi \\ &= (\hat{f} \cdot D_{a^{-1}} \overline{\hat{\psi}})^\vee (ab). \end{aligned}$$

The Besov spaces $B_{p,q}^\alpha(\mathbb{R})$, where $\alpha > 0$ and $1 \le p, q \le \infty$, are the natural function spaces associated with the CWT, namely, their norms quantify time-scale concentration of functions or distributions. They consist of functions in $L^p(\mathbb{R})$ with "smoothness α," with the parameter q allowing for a finer graduation of the quantification of smoothness. There are many equivalent definitions of the Besov spaces, and we refer to Triebel [122] for more information. An important fact is that equivalent norms for the Besov spaces can be formulated in terms of the discrete wavelet transform (see Meyer [102]) or the continuous wavelet transform (see Perrier and Basdevant [106]).

For practical purposes, however, discrete wavelet systems are needed, i.e., wavelet systems $\{\sigma(a,b)\psi\}_{(a,b)\in\Lambda}$, where Λ does not equal $\mathbb{A}$, but instead is just a sequence in $\mathbb{A}$. We remark that although Λ will always denote a countable sequence of points in $\mathbb{A}$ and not merely a subset, for simplicity we will write $\Lambda \subseteq \mathbb{A}$. In particular, this means that we allow repetitions of points. Further recall that the *disjoint union* $S = \bigcup_{i=1}^{n} S_i$ of a finite collection of sequences $S_1, \ldots, S_n$ contained in some set is the sequence $S = \{s_{11}, \ldots, s_{1n}, s_{21}, \ldots, s_{2n}, \ldots\}$, where each S_i is indexed as $S_i = \{s_{ki}\}_{k \in \mathbb{N}}$, i.e., S is the sequence obtained by amalgamating $S_1, \ldots, S_n$.

Definition 2.3. *(a) Given an analyzing wavelet $\psi \in L^2(\mathbb{R})$, a sequence of* time-scale indices $\Lambda \subseteq \mathbb{A}$*, and a weight function $w : \Lambda \to \mathbb{R}^+$, the* weighted (irregular) wavelet system *generated by ψ, Λ, and w is defined by*

$$\begin{aligned}\mathcal{W}(\psi, \Lambda, w) &= \{w(a,b)^{\frac{1}{2}}\, \sigma(a,b)\psi\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, D_a T_b \psi\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, \tfrac{1}{\sqrt{a}} \psi(\tfrac{x}{a} - b)\}_{(a,b)\in\Lambda}.\end{aligned}$$

If $w = 1$ we omit writing it.

(b) Let $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$, and let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the weighted (irregular) wavelet system *generated by $\{(\psi_\ell, \Lambda_\ell, w_\ell)\}_{\ell=1}^{L}$ is the disjoint union*

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell).$$

This definition of weighted wavelet systems includes as special cases the classical affine systems, the quasi-affine systems, and the co-affine systems (defined below). In particular, it is important to allow the case of nonconstant weights in order to obtain the quasi-affine systems.

The most often employed and studied wavelet systems are the *classical affine systems*

$$\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k\in\mathbb{Z}}),$$

where $\psi \in L^2(\mathbb{R})$ and $a > 1$, $b > 0$. Since these systems lack the property of being shift-invariant, i.e., of being invariant under integer translations, the so-called *quasi-affine systems*

$$\mathcal{W}(\psi, \{(a^j, bk)\}_{j<0,k\in\mathbb{Z}} \cup \{(a^j, a^{-j}bk)\}_{j\geq 0,k\in\mathbb{Z}}, w),$$

where

$$\begin{aligned} w(a^j, bk) &= 1, \qquad && j < 0,\ k \in \mathbb{Z}, \\ w(a^j, a^{-j}bk) &= a^{-j}, \qquad && j \geq 0,\ k \in \mathbb{Z}, \end{aligned}$$

were developed for $a \in \mathbb{Z}$, $b > 0$ by Ron and Shen [109] and for $a \in \mathbb{Q}$ by Bownik [11]. Further contributions werde made by Chui, Shi, and Stöckler [35], Gressman, Labate, Weiss, and Wilson [61], and Johnson [88]. In [61] Gressman, Labate, Weiss, and Wilson also studied classical affine systems for

$b = 1$ with interchanged ordering of dilation and translation, i.e., wavelet systems of the form

$$\{T_k D_{a^j} \psi\}_{j,k\in\mathbb{Z}} = \{D_{a^j} T_{a^{-j}k} \psi\}_{j,k\in\mathbb{Z}},$$

where $a > 1$. This amounts, in the terminology of this book and letting $b > 0$ be arbitrary, to taking

$$\mathcal{W}(\psi, \{(a^j, a^{-j}bk)\}_{j,k\in\mathbb{Z}}).$$

These are the so-called *co-affine systems*. We also refer to Johnson [90].

Recently a general notion of oversampled affine systems was introduced by Hernández, Labate, Weiss, and Wilson [78] and extended by Johnson [89], which includes not only the classical affine, but also the quasi-affine and co-affine systems as special cases.

Definition 2.4. *Given $\psi \in L^2(\mathbb{R})$, $a > 1$, $b > 0$, and $\{r_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{R}^+$, an* oversampled affine system *is a weighted wavelet system of the form $\mathcal{W}(\psi, \Lambda, w)$ with*

$$\Lambda = \left\{(a^j, \tfrac{bk}{r_j})\right\}_{j,k\in\mathbb{Z}} \qquad \text{and} \qquad w(a^j, \tfrac{bk}{r_j}) = \tfrac{1}{r_j}.$$

Example 2.5. The following are special cases of oversampled affine systems.

(i) The classical affine systems are obtained by setting $r_j \equiv 1$.

(ii) The quasi-affine systems of Ron and Shen [109] are obtained when a is an integer, $b = 1$, and

$$r_j = \begin{cases} 1, & j < 0, \\ a^j, & j \geq 0. \end{cases}$$

(iii) The quasi-affine systems of Bownik [11] are obtained when $a = \frac{p}{q}$ is rational, $b = 1$, and

$$r_j = \begin{cases} q^{-j}, & j < 0, \\ p^j, & j \geq 0. \end{cases}$$

(iv) The co-affine systems of Gressman, Labate, Weiss, and Wilson [61] are obtained by setting $r_j = a^j$ and $b = 1$.

2.3 Time-Frequency Analysis

As in the section before, we will state the basic definitions, notations, and results from time-frequency analysis as far as we will need them later. We mention the books by Daubechies [41], Feichtinger and Strohmer [56, 57], and Gröchenig [63] as references for further details.

In time-frequency analysis, time-frequency shifts play the role that time-scale shifts play in the wavelet setting. The time-frequency plane is actually the Heisenberg group $\mathbb{H} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ endowed with the multiplication

$$(a, b, c)(a', b', c') = (a + a', b + b', cc' e^{-2\pi i \langle a, b' \rangle}).$$

The toral component will later be ignored. This group is equipped with the so-called Schrödinger representation, which is the irreducible unitary representation of $\mathbb{H}$ on $L^2(\mathbb{R}^d)$ defined by

$$(\rho(a, b, c)g)(x) = c\, e^{2\pi i \langle b, x \rangle} g(x - a) = cM_b T_a g(x),$$

where M_b denotes the *modulation operator* $M_b f(x) = e^{2\pi i \langle b, x \rangle} f(x)$.

The analogue of the continuous wavelet transform is the *Short-Time Fourier transform (STFT)* of $f \in L^2(\mathbb{R}^d)$ with respect to $g \in L^2(\mathbb{R}^d)$ given by

$$V_g f(a, b) = \langle f, \rho(a, b, 1)g \rangle = \langle f, M_b T_a g \rangle = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \langle b, x \rangle} \overline{g(x - a)}\, dx$$

for $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$. We have

$$|V_g f(a, b)| \le \|f\|_2 \, \|g\|_2 \quad \text{for all } (a, b) \in \mathbb{R}^{2d}$$

and $V_g f \in C(\mathbb{R}^{2d})$. Unlike the wavelet case, all vectors in $L^2(\mathbb{R})$ are admissible, because the Haar measure on the Heisenberg group is unimodular. The STFT V_g maps $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ for all $g \in L^2(\mathbb{R}^d)$. Given $g_1, g_2 \in L^2(\mathbb{R}^d)$ and $f_1, f_2 \in L^2(\mathbb{R}^d)$, the *orthogonality relations* for the STFT are

$$\left\langle V_{g_1} f_1,\, V_{g_2} f_2 \right\rangle = C_{g_1, g_2} \left\langle f_1, f_2 \right\rangle, \tag{2.4}$$

where $C_{g_1, g_2} = \int_{-\infty}^{\infty} \overline{g_1(x)}\, g_2(x)\, dx$. There is a great deal of symmetry between g and f in the STFT; more precisely, $V_f g(a, b) = e^{-2\pi i \langle a, b \rangle} \overline{V_g f(-a, -b)}$, hence, in particular, $\|V_g f\|_2 = \|V_f g\|_2$.

The modulation spaces are the natural function spaces associated with the STFT, namely, their norms quantify time-frequency concentration of functions or distributions in the same way that the Besov space norms quantify time-scale concentration. In particular, the *modulation space* $M^1(\mathbb{R}^d)$ consists of all functions $f \in L^1(\mathbb{R}^d)$ for which the following norm is finite:

$$\|f\|_{M^1(\mathbb{R}^d)} = \|V_g f\|_{L^1(\mathbb{R}^{2d})} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g f(a, b)|\, db\, da,$$

where g is any nonzero Schwartz-class function (each choice of g yields the same space under an equivalent norm). This modulation space was first defined by Feichtinger in [47] and is therefore also called the *Feichtinger algebra* (and is sometimes denoted by S_0); it is an algebra under both pointwise multiplication and convolution, and has many other remarkable properties. Notice that $M^1(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. Moreover, we have the following well-known result on the relation between integrable STFTs and generators being in $M^1(\mathbb{R}^d)$.

Proposition 2.6. *The following conditions are equivalent.*

(i) $f \in M^1(\mathbb{R}^d)$.
(ii) $V_f f \in L^1(\mathbb{R}^{2d})$.

For more details on modulation spaces we refer the reader to Gröchenig [63].

As in wavelet theory discrete versions of the continuous Gabor systems are of considerable interest. Notice that also in this case, although Λ will always denote a sequence of points in $\mathbb{R}^d$ and not merely a subset, for simplicity we will write $\Lambda \subseteq \mathbb{R}^d$. Recall the definition of a disjoint union of sequences from Section 2.2.

Definition 2.7. *(a) Given a generator* $g \in L^2(\mathbb{R})$*, a sequence of* time-frequency indices $\Lambda \subseteq \mathbb{R}^{2d}$*, and a weight function* $w : \Lambda \to \mathbb{R}^+$*, the* weighted (irregular) Gabor system *generated by* g*,* Λ*, and* w *is defined by*

$$\begin{aligned}\mathcal{G}(g, \Lambda, w) &= \{w(a,b)^{\frac{1}{2}}\, \rho(a,b,1)g\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, M_b T_a g\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, e^{2\pi i\langle b,x\rangle} g(x-a)\}_{(a,b)\in\Lambda}.\end{aligned}$$

If $w = 1$ *we omit writing it.*

(b) Let $g_1, \ldots, g_L \in L^2(\mathbb{R}^d)$*, and let* $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{R}^{2d}$ *with associated weight functions* $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ *for* $\ell = 1, \ldots, L$ *be given. Then the* weighted (irregular) Gabor system *generated by* $\{(g_\ell, \Lambda_\ell, w_\ell)\}_{\ell=1}^L$ *is the disjoint union*

$$\bigcup_{\ell=1}^{L} \mathcal{G}(g_\ell, \Lambda_\ell, w_\ell).$$

This definition of weighted Gabor systems includes as special cases the so-called *regular Gabor systems*, which are Gabor systems of the form $\mathcal{G}(g, a\mathbb{Z}^d \times b\mathbb{Z}^d)$, where $a, b > 0$.

One tool for studying irregular nonweighted Gabor systems is the notion of Beurling density. For $h > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we let $Q_h(x)$ denote the cube centered at x with side length h, i.e., $Q_h(x) = \prod_{j=1}^d \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right)$. Then the *upper Beurling density* of a sequence Λ in $\mathbb{R}^d$ is defined by

$$\mathcal{D}^+(\Lambda) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d},$$

and its *lower Beurling density* is

$$\mathcal{D}^-(\Lambda) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d}.$$

If $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$, then we say that Λ has *uniform Beurling density* and denote this density by $\mathcal{D}(\Lambda)$. For example, the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \subseteq \mathbb{R}^2$, $a, b > 0$ has the uniform Beurling density $\mathcal{D}(\Lambda) = \frac{1}{ab}$.

It was shown by H. Landau [97, Lem. 4], that these densities do not depend on the particular choice of sets $Q_h(x)$, $h > 0$, $x \in \mathbb{R}^d$, in the sense that we can substitute these sets by sets $x + hU$, where $U \subseteq \mathbb{R}^d$ is a compact set with $|U| = 1$, i.e., of Lebesgue measure 1, whose boundary has measure zero, yet still obtain the same notion.

For more details on Beurling density and its connections to Gabor frames we refer the reader to the papers authored by Balan, Casazza, Heil, and Z. Landau [7] and by Christensen, Deng, and Heil [22]. New characterizations and an extension of the notion of Beurling density to weighted sequences can be found in Kutyniok [93]. This paper also contains a fundamental relationship between this density, the frame bounds, and the norm of the generator for weighted Gabor frames.

2.4 Amalgam Spaces

An amalgam space combines a local criterion for membership with a global criterion. The first amalgam spaces were introduced by Wiener in his study of generalized harmonic analysis [125, 126]. A comprehensive general theory of amalgam spaces on locally compact groups was introduced and extensively studied by Feichtinger and Gröchenig, e.g., [48, 52, 53, 50]. For an expository introduction to Wiener amalgams on $\mathbb{R}$ with extensive references to the original literature, we refer to Heil [71]. In the following we will give a brief survey of amalgam spaces of the type $W_G(L^\infty, L^p)$ and $W_G(C, L^p)$, where $1 \leq p < \infty$ and G is a locally compact group. For the theory of locally compact groups we refer the reader to Folland [58] and Hewitt and Ross [80, 81].

Let G be a locally compact group and let μ_G denote a left-invariant Haar measure on G. Then the Wiener amalgam spaces $W_G(L^\infty, L^p)$ and $W_G(C, L^p)$ are defined as follows.

Definition 2.8. *Given* $1 \leq p < \infty$, *the* amalgam space $W_G(L^\infty, L^p)$ *on the locally compact group* G *consists of all functions* $f\colon G \to \mathbb{C}$ *such that*

$$\|f\|_{W_G(L^\infty,L^p)} = \left(\int_G \operatorname{ess\,sup}_{a\in G} |f(a)\, \phi(x^{-1}a))|^p \, d\mu_G(x) \right)^{1/p} < \infty,$$

where ϕ *is a fixed continuous function with compact support satisfying* $0 \leq \phi(x) \leq 1$ *for all* $x \in G$, *and* $\phi(x) = 1$ *on some compact neighborhood of the identity. The amalgam space* $W_G(C, L^p)$ *is the closed subspace of* $W_G(L^\infty, L^p)$ *consisting of the continuous functions in* $W_G(L^\infty, L^p)$.

$W_G(L^\infty, L^p)$ is a Banach space, and its definition is independent of the choice of ϕ, in the sense that each choice of ϕ yields the same space under an equivalent norm. For proofs and more details, see Feichtinger and Gröchenig [52, 53].

The space $W_G(L^\infty, L^p)$ can be equipped with an equivalent discrete-type norm. For this, we first require some notation. Given some neighborhood U of the identity in G, a sequence $\{x_i\}_{i\in I}$ in G is called *U-dense*, if $\bigcup_{i\in I} x_i U = G$. It is called *$V$-separated*, if for some relatively compact neighborhood V of the identity the sets $\{x_i V\}_{i\in I}$ are pairwise disjoint. The sequence is called *relatively separated*, if it is the finite union of V-separated sequences.

Definition 2.9. *A sequence of continuous functions $\{\phi_i\}_{i\in I}$ on G is called a* bounded partition of unity, *or* BUPU, *if*

(i) $0 \leq \phi_i(x) \leq 1$ *for all $i \in I$ and $x \in G$.*
(ii) $\sum_{i\in I} \phi_i \equiv 1$.
(iii) *There exists a compact neighborhood U of the identity in G with nonempty interior and a U-dense, relatively separated sequence $\{x_i\}_{i\in I}$ such that* $\operatorname{supp}(\phi_i) \subseteq x_i U$ *for all $i \in I$.*

Then we have the following result from Feichtinger [49] (compare also Feichtinger and Gröchenig [52]).

Theorem 2.10. *If $\{\phi_i\}_{i\in I}$ is a BUPU, then*

$$\|f\|_{W_G(L^\infty,L^p)} \asymp \left(\sum_{i\in I} \|f\phi_i\|_\infty^p\right)^{\frac{1}{p}},$$

where $\asymp$ denotes the equivalence of norms.

3

Weighted Affine Density

In this chapter we introduce the notion of weighted affine density for multiple weighted sequences in the affine group. We study its basic properties, which will be used in the subsequent chapters. Then we discuss the notion of affine density introduced by Sun and Zhou [119] and study the exact relation to the affine density we will employ in this book.

This chapter contains a generalization of the notion of weighted affine density as introduced by Heil and Kutyniok in [73] to multiple weighted sequences. Further, some of the results in this chapter generalize the results obtained in Kutyniok [92] to multiple weighted sequences.

3.1 Definitions

The notion of affine density, which we will introduce in this section, will employ the affine group $\mathbb{A}$ defined in Section 2.2. The density defined by Sun and Zhou [119], which will be studied in Section 3.3 is suited to the geometry of an isomorphic version of the affine group. Surprisingly, it turns out that both notions of density behave very differently.

In $\mathbb{R}^d$, Beurling density is a measure of the "average" number of points of a sequence that lie inside a unit cube. We will transfer the definition of Beurling density for sequences in $\mathbb{R}^2$ to sequences in the affine group and extend it in order to allow multiple weighted sequences.

Before we can state the definition of affine density, we first require some notation. For $h > 0$, we let Q_h denote a fixed family of increasing, exhaustive neighborhoods of the identity element $e = (1, 0)$ in $\mathbb{A}$. For simplicity, we will take

$$Q_h = [e^{-\frac{h}{2}}, e^{\frac{h}{2}}) \times [-\tfrac{h}{2}, \tfrac{h}{2}).$$

For $(x, y) \in \mathbb{A}$, we let $Q_h(x, y)$ be the set Q_h left-translated via the group action so that it is "centered" at the point (x, y), i.e.,

$$Q_h(x,y) = (x,y)\cdot Q_h = \big\{(xa, \tfrac{y}{a}+b) : a \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}}),\, b \in [-\tfrac{h}{2}, \tfrac{h}{2})\big\}.$$

We remark that although Q_h is a rectangle, the sets $Q_h(x,y)$ do not share this property. However, for simplicity we will still refer to the sets $Q_h(x,y)$ as "boxes".

A locally compact group is always equipped with a left-invariant Haar measure, which is unique up to a constant multiple. For $\mathbb{A}$ such a measure is given by $\mu_{\mathbb{A}} = \frac{dx}{x}dy$. Since $\mu_{\mathbb{A}}$ is left-invariant, we have that

$$\mu_{\mathbb{A}}(Q_h(x,y)) = \mu_{\mathbb{A}}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{dx}{x}\, dy = h^2.$$

Next, given a sequence Λ in $\mathbb{A}$ and a weight function $w : \Lambda \to \mathbb{R}^+$, we define the weighted number of elements of Λ lying in a subset K of Λ to be

$$\#_w(K) = \sum_{(a,b)\in K} w(a,b).$$

A sequence Λ together with its associated weight function w will always be denoted by (Λ, w).

Now we can state the definition of affine density for multiple weighted sequences $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$.

Definition 3.1. *Let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the* upper weighted affine density *of $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ is defined by*

$$\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\sum_{\ell=1}^L \#_{w_\ell}(\Lambda_\ell \cap Q_h(x,y))}{h^2},$$

and the lower weighted affine density *of $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ is*

$$\mathcal{D}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \liminf_{h\to\infty} \inf_{(x,y)\in\mathbb{A}} \frac{\sum_{\ell=1}^L \#_{w_\ell}(\Lambda_\ell \cap Q_h(x,y))}{h^2}.$$

If $\mathcal{D}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L)$, then we say that $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ has uniform weighted affine density *and denote this density by $\mathcal{D}(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L)$. If $w_1 = \ldots = w_L = 1$, we write $\mathcal{D}^-(\{\Lambda_\ell\}_{\ell=1}^L)$ and $\mathcal{D}^+(\{\Lambda_\ell\}_{\ell=1}^L)$.*

We first make the following basic observations.

Remark 3.2. (a) Let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ be given. The definition of affine density for multiple sequences can be reduced to a simpler form using the disjoint union $\Lambda = \bigcup_{\ell=1}^L \Lambda_\ell$ of the sequences $\Lambda_1, \ldots, \Lambda_L$. Employing this new sequence we obtain

$$\mathcal{D}^+(\{\Lambda_\ell\}_{\ell=1}^L) = \mathcal{D}^+(\Lambda) \quad \text{and} \quad \mathcal{D}^-(\{\Lambda_\ell\}_{\ell=1}^L) = \mathcal{D}^-(\Lambda).$$

(b) Let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then

$$\sum_{\ell=1}^{L} \mathcal{D}^-(\Lambda_\ell, w_\ell) \leq \mathcal{D}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) \leq \mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) \leq \sum_{\ell=1}^{L} \mathcal{D}^+(\Lambda_\ell, w_\ell).$$

These inequalities may be strict, for instance, consider $\Lambda_1 = \{(2^j, k)\}_{j \geq 0, k \in \mathbb{Z}}$ and $\Lambda_2 = \{(2^j, k)\}_{j<0, k \in \mathbb{Z}}$ and $w_1 = w_2 = 1$, where $L = 2$.

The best known class of wavelet systems are the classical affine systems, which are $\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k \in \mathbb{Z}})$ with $a > 1, b > 0$, and $\psi \in L^2(\mathbb{R})$. For Gabor systems $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ with $a, b > 0$, the Beurling density $\frac{1}{ab}$ of the lattice $a\mathbb{Z} \times b\mathbb{Z}$ is a ubiquitous constant in a variety of formulas. For example, if $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ is a tight frame for $L^2(\mathbb{R})$ and $\|g\|_2 = 1$, then the frame bounds are exactly $\frac{1}{ab}$. Many of these formulas have analogues for classical affine systems, with the number $\frac{1}{b \ln a}$ playing the role that $\frac{1}{ab}$ plays for Gabor systems. For this reason, Daubechies already suggested in [41, Sec. 4.1], that $\frac{1}{b \ln a}$ might play the role of a density for affine systems, but she also demonstrated that affine systems cannot possess an analogue of the Nyquist density that Gabor systems possess. The following result shows that our notion of density in fact leads to exactly this value. In Section 4.3 it will be shown that this is a special case of a larger class of wavelet systems, the oversampled affine systems, which surprisingly all possess the same uniform weighted affine density $\frac{1}{b \ln a}$.

Lemma 3.3. *Let $a > 1$, $b > 0$, and define $\Lambda = \{(a^j, bk)\}_{j,k \in \mathbb{Z}}$. Then Λ has uniform affine density*

$$\mathcal{D}(\Lambda) = \frac{1}{b \ln a}.$$

Proof. Fix $(x, y) \in \mathbb{A}$. If $(a^j, bk) \in Q_h(x, y)$, then

$$\left(\tfrac{a^j}{x}, bk - \tfrac{xy}{a^j}\right) = (x, y)^{-1} \cdot (a^j, bk) \in Q_h.$$

In particular, $\frac{a^j}{x} \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$. There are at least $\frac{h}{\ln a}$ and at most $\frac{h}{\ln a} + 1$ integers j satisfying this condition. Additionally, we have $\frac{xy}{a^j b} - \frac{h}{2b} \leq k < \frac{xy}{a^j b} + \frac{h}{2b}$. For a given j, there are at least $\frac{h}{b}$ and at most $\frac{h}{b} + 1$ integers k satisfying this condition. We conclude that

$$\tfrac{h}{\ln a} \cdot \tfrac{h}{b} \leq \#(\Lambda \cap Q_h(x, y)) \leq \left(\tfrac{h}{\ln a} + 1\right)\left(\tfrac{h}{b} + 1\right).$$

Thus

$$\mathcal{D}^+(\Lambda) \leq \limsup_{h \to \infty} \frac{(\frac{h}{\ln a} + 1)(\frac{h}{b} + 1)}{h^2} = \frac{1}{b \ln a},$$

and similarly $\mathcal{D}^-(\Lambda) \geq \frac{1}{b \ln a}$. □

In Section 3.3 it will be shown that employing a different, yet isomorphic version of the affine group yields a different value for the affine density of classical affine systems.

3.2 Basic Properties

The following proposition, whose proof will be given at the end of this section, indicates that provided we are only interested in "qualitative" values of upper affine density, it suffices to consider the single weighted sequences separately. We further show that the multiple weighted sequences have uniform affine density if this is true for each of the single weighted sequences.

Proposition 3.4. *Let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the following statements hold.*

(i) *We have*

$$\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty \iff \mathcal{D}^+(\Lambda_\ell, w_\ell) < \infty \text{ for all } \ell = 1, \ldots, L.$$

(ii) *If for each $\ell = 1, \ldots, L$ the pair (Λ_ℓ, w_ℓ) has uniform affine density, then also $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ has uniform affine density.*

Thus we now focus on single weighted sequences in $\mathbb{A}$. We will derive some equivalent ways to view the meaning of finite upper weighted affine density and positive lower weighted affine density. First, however, we will require the following technical lemma, which will be used throughout. It studies the number of overlaps of boxes of the form $Q_h(x, y)$. In particular, we derive a covering of the affine group with special boxes, however not a disjoint one.

Lemma 3.5. *Let $h > 0$, $r \geq 1$, and $(p, q) \in \mathbb{A}$ be given.*

(i) $\left\{Q_h(e^{jh}, khe^{-\frac{h}{2}}) \cdot (p, q)\right\}_{j,k\in\mathbb{Z}}$ *covers* $\mathbb{A}$.

(ii) *Any set $Q_h(x, y)$ intersects at most $3(e^h(p+1) + e^{\frac{h}{2}}p|q| + 1)$ different sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}}) \cdot (p, q)$.*

(iii) *Any set $Q_{rh}(x, y)$ intersects at most $(r + 2)(e^h(r + 1) + 1)$ different sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$.*

(iv) *Any set $Q_{rh}(x, y)$ contains at least $(r - 1)(\frac{r-1}{e^h} - 1)$ disjoint sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$.*

Proof. (i) Since $\mathbb{A}$ is invariant under right-shifts, it suffices for this part to consider the case $(p, q) = (1, 0)$. Fix any $(x, y) \in \mathbb{A}$. Then there is a unique $j \in \mathbb{Z}$ and $a \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$ such that $x = e^{jh}a$, or $\ln x = jh + \ln a$. Further, since $\frac{h}{a}e^{-\frac{h}{2}} \leq h$, there exist at least one $k \in \mathbb{Z}$ and $b \in [-\frac{h}{2}, \frac{h}{2})$ such that $y = \frac{kh}{a}e^{-\frac{h}{2}} + b$. Consequently,

$$(x, y) = (e^{jh}a, \tfrac{kh}{a}e^{-\frac{h}{2}} + b) = (e^{jh}, khe^{-\frac{h}{2}}) \cdot (a, b) \in Q_h(e^{jh}, khe^{-\frac{h}{2}}).$$

(ii) Fix $(x, y) \in \mathbb{A}$, and suppose that $(u, v) \in Q_h(x, y) \cap Q_h(e^{jh}, khe^{-\frac{h}{2}}) \cdot (p, q)$. Then there exist points (a, b), $(c, d) \in Q_h$ such that

$$(u,v) = (x,y)\cdot(a,b) = (ax, \tfrac{y}{a}+b) \in Q_h(x,y)$$

and

$$\begin{aligned}(u,v) &= (e^{jh}, khe^{-\frac{h}{2}})\cdot(c,d)\cdot(p,q)\\ &= (e^{jh}cp, \tfrac{kh}{cp}e^{-\frac{h}{2}} + \tfrac{d}{p} + q) \in Q_h(e^{jh}, khe^{-\frac{h}{2}})\cdot(p,q).\end{aligned}$$

In particular, $\frac{ax}{cp} = e^{jh}$ with $e^{-\frac{h}{2}} \le a, c < e^{\frac{h}{2}}$, so $\frac{x}{e^h p} \le e^{jh} \le \frac{e^h x}{p}$. Therefore

$$\frac{1}{h}(\ln x - h - \ln p) \le j \le \frac{1}{h}(\ln x + h - \ln p),$$

which is satisfied for at most 3 values of j. Further, $khe^{-\frac{h}{2}} = \frac{xy}{e^{jh}} + bcp - cd - cpq$, so

$$\frac{e^{\frac{h}{2}}}{h}\left(\frac{xy}{e^{jh}} - \frac{h}{2}e^{\frac{h}{2}}p - \frac{h}{2}e^{\frac{h}{2}} - \frac{h}{2}p|q|\right) \le k \le \frac{e^{\frac{h}{2}}}{h}\left(\frac{xy}{e^{jh}} + \frac{h}{2}e^{\frac{h}{2}}p + \frac{h}{2}e^{\frac{h}{2}} + \frac{h}{2}p|q|\right).$$

For a given value of j, this is satisfied for at most $e^h(p+1) + e^{\frac{h}{2}}p|q| + 1$ values of k. Thus, $Q_h(x,y)$ can intersect at most $3(e^h(p+1) + e^{\frac{h}{2}}p|q| + 1)$ sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$.

(iii), (iv) The proofs are similar to the proof of part (ii). □

Using this lemma, we can give a useful reinterpretation of finite upper weighted affine density, which is inspired by a result for (nonweighted) Beurling density by Christensen, Deng, and Heil [22, Lem. 2.3].

Proposition 3.6. *If $\Lambda \subseteq \mathbb{A}$ and $w : \Lambda \to \mathbb{R}^+$, then the following conditions are equivalent.*

(i) $\mathcal{D}^+(\Lambda, w) < \infty$.

(ii) *There exists $h > 0$ such that* $\sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) < \infty$.

(iii) *For every $h > 0$ we have* $\sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) < \infty$.

Proof. (i) ⇒ (ii) and (iii) ⇒ (ii) are trivial.

(ii) ⇒ (i), (iii). Suppose there exists $h > 0$ such that

$$M = \sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) < \infty.$$

For $1 < t < h$, we have $Q_t(x,y) \subseteq Q_h(x,y)$, so $\sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_t(x,y)) < \infty$. On the other hand, if $t \ge h$ then we have $t = rh$ with $r \ge 1$. If we let $N_r = (r+2)(e^h(r+1)+1)$ be as given in Lemma 3.5(iii), then each set $Q_{rh}(x,y)$ is covered by a union of at most N_r sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$. Consequently,

$$\sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_{rh}(x,y)) \le N_r \sup_{j,k\in\mathbb{Z}} \#_w(\Lambda \cap Q_h(e^{jh}, khe^{-\frac{h}{2}})) \le N_r M < \infty.$$

Thus statement (iii) holds. Further,

$$\mathcal{D}^+(\Lambda, w) \le \limsup_{r\to\infty} \frac{N_r M}{(rh)^2} = \frac{Me^h}{h^2} < \infty,$$

so statement (i) holds as well. □

A similar result holds for the case of positive lower weighted affine density.

Proposition 3.7. *If* $\Lambda \subseteq \mathbb{A}$ *and* $w : \Lambda \to \mathbb{R}^+$, *then the following conditions are equivalent.*

(i) $\mathcal{D}^-(\Lambda, w) > 0$.

(ii) *There exists some* $h > 0$ *such that* $\inf_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) > 0$.

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (i). Suppose there exists $h > 0$ such that $M = \inf_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) > 0$. Let $r > 1$. If we let $N_r = (r-1)(\frac{r-1}{e^h} - 1)$ be as given in Lemma 3.5(iv), then each set $Q_{rh}(x,y)$ contains at least N_r disjoint sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$. Consequently,

$$\inf_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_{rh}(x,y)) \ge N_r \inf_{j,k\in\mathbb{Z}} \#_w(\Lambda \cap Q_h(e^{jh}, khe^{-\frac{h}{2}})) \ge N_r M > 0.$$

Therefore,

$$\mathcal{D}^-(\Lambda, w) \ge \liminf_{r\to\infty} \frac{N_r M}{(rh)^2} = \frac{Me^{-h}}{h^2} > 0,$$

which is (i). □

We finish this section by giving the proof of Proposition 3.4.

Proof (of Proposition 3.4). (i) Assume $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty$ and fix some $\ell \in \{1, \ldots, L\}$. Then, by definition, there exist $h > 0$ and $N < \infty$ such that $\#_{w_\ell}(\Lambda_\ell \cap Q_h(x,y)) < N$ for all $(x,y) \in \mathbb{A}$. By Proposition 3.6, this implies $\mathcal{D}^+(\Lambda_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$.

Conversely, if $\mathcal{D}^+(\Lambda_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$, then, by Remark 3.2(b),

$$\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) \le \sum_{\ell=1}^L \mathcal{D}^+(\Lambda_\ell, w_\ell) < \infty.$$

(ii) Suppose that $\mathcal{D}^-(\Lambda_\ell, w_\ell) = \mathcal{D}^+(\Lambda_\ell, w_\ell) = \mathcal{D}(\Lambda_\ell, w_\ell)$ for all $\ell = 1, \ldots, L$. By Remark 3.2(b), this implies

$$\sum_{\ell=1}^{L} \mathcal{D}(\Lambda_\ell, w_\ell) \leq \mathcal{D}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^{L}) \leq \mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^{L}) \leq \sum_{\ell=1}^{L} \mathcal{D}(\Lambda_\ell, w_\ell),$$

hence

$$\mathcal{D}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^{L}) = \mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^{L}) = \sum_{\ell=1}^{L} \mathcal{D}(\Lambda_\ell, w_\ell).$$

This proves (ii). □

3.3 The Notion of Affine Density by Sun and Zhou

As already mention in Section 3.1, the notion of affine density can also be defined by employing a group isomorphic to $\mathbb{A}$. Let $\widetilde{\mathbb{A}} = \mathbb{R}^+ \times \mathbb{R}$ denote this group, which is endowed with multiplication given by

$$(a, b) \star (x, y) = (ax, b + ay).$$

To distinguish this group multiplication from the one associated with the group $\mathbb{A}$, we use the operator sign $\star$ here. The identity element of $\widetilde{\mathbb{A}}$ is $e = (1, 0)$, and the inverse of an element $(a, b) \in \widetilde{\mathbb{A}}$ is given by $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$.

To construct a wavelet system adapted to this group, let $\tilde{\sigma}$ be the unitary representation of $\widetilde{\mathbb{A}}$ on $L^2(\mathbb{R})$ defined by

$$(\tilde{\sigma}(a, b)\psi)(x) = \tfrac{1}{\sqrt{a}}\psi(\tfrac{x-b}{a}).$$

Given a function $\psi \in L^2(\mathbb{R})$, a sequence $\Lambda \subseteq \widetilde{\mathbb{A}}$, and a weight function $w : \Lambda \to \mathbb{R}^+$, the *weighted (irregular) wavelet system* generated by ψ, Λ, and w is defined by

$$\begin{aligned} \widetilde{\mathcal{W}}(\psi, \Lambda, w) &= \{w(a, b)^{\frac{1}{2}}\, \tilde{\sigma}(a, b)\psi\}_{(a,b)\in\Lambda} \\ &= \{w(a, b)^{\frac{1}{2}}\, \tfrac{1}{\sqrt{a}}\psi(\tfrac{x-b}{a})\}_{(a,b)\in\Lambda} \\ &= \{w(a, b)^{\frac{1}{2}}\, T_b D_a \psi\}_{(a,b)\in\Lambda}. \end{aligned}$$

Given $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ and $\Lambda_1, \ldots, \Lambda_L \subseteq \widetilde{\mathbb{A}}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, the *weighted (irregular) wavelet system* generated by $\{(\psi_\ell, \Lambda_\ell, w_\ell)\}_{\ell=1}^{L}$ is the disjoint union

$$\bigcup_{\ell=1}^{L} \widetilde{\mathcal{W}}(\psi_\ell, \Lambda_\ell, w_\ell).$$

Now density with respect to the geometry of $\widetilde{\mathbb{A}}$ can be defined in a similar way as in Section 3.1. Recall that the boxes Q_h were chosen to be $Q_h =$

$[e^{-\frac{h}{2}}, e^{\frac{h}{2}}) \times [-\frac{h}{2}, \frac{h}{2})$. Then, for $h > 0$ and $(x, y) \in \widetilde{\mathbb{A}}$, we define the boxes $\widetilde{Q}_h(x, y)$ by

$$\widetilde{Q}_h(x,y) = (x,y) \star Q_h = \{(xa, y + xb) : a \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}}),\ b \in [-\tfrac{h}{2}, \tfrac{h}{2})\}$$

We remark that in this situation the sets $\widetilde{Q}_h(x, y)$ remain rectangles, thus the geometry of the measuring boxes is easier in this sense.

A left-invariant Haar measure for the group $\widetilde{\mathbb{A}}$ is given by $\mu_{\widetilde{\mathbb{A}}} = \frac{dx}{x^2} dy$. Hence the volume of a box $\widetilde{Q}_h(x, y)$ equals

$$\mu_{\widetilde{\mathbb{A}}}(\widetilde{Q}_h(x,y)) = \mu_{\widetilde{\mathbb{A}}}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{dx}{x^2} dy = h(e^{\frac{h}{2}} - e^{-\frac{h}{2}}).$$

The definition of weighted affine density with respect to the group $\widetilde{\mathbb{A}}$ is then as follows. Let $\Lambda_1, \ldots, \Lambda_L \subseteq \widetilde{\mathbb{A}}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the *upper weighted affine density with respect to* $\widetilde{\mathbb{A}}$ of $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ is defined by

$$\widetilde{\mathcal{D}}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \limsup_{h\to\infty} \sup_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\sum_{\ell=1}^L \#_{w_\ell}(\Lambda_\ell \cap \widetilde{Q}_h(x,y))}{h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})},$$

and the *lower weighted affine density with respect to* $\widetilde{\mathbb{A}}$ of $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ is

$$\widetilde{\mathcal{D}}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \liminf_{h\to\infty} \inf_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\sum_{\ell=1}^L \#_{w_\ell}(\Lambda_\ell \cap \widetilde{Q}_h(x,y))}{h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})}.$$

If $\widetilde{\mathcal{D}}^-(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \widetilde{\mathcal{D}}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L)$, then we say that $\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L$ has *uniform weighted affine density with respect to* $\widetilde{\mathbb{A}}$ and denote this density by $\widetilde{\mathcal{D}}(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L)$. If $w_1 = \ldots = w_L = 1$, we omit writing it.

In contrast to Lemma 3.5, the sets $\widetilde{Q}_h(x, y)$ can be chosen to form a *disjoint* covering of $\widetilde{\mathbb{A}}$.

Lemma 3.8. *Let $h > 0$ and $r \geq 1$ be given.*

(i) $\{\widetilde{Q}_h(e^{jh}, khe^{jh})\}_{j,k\in\mathbb{Z}}$ *is a disjoint covering of* $\widetilde{\mathbb{A}}$.

(ii) *Any set $\widetilde{Q}_{rh}(x, y)$ intersects at most* $\frac{r(e^{h(\frac{r+5}{2})} - e^{-h(\frac{r+3}{2})})}{e^h - 1} + 2(r + 2)$ *sets of the form* $\widetilde{Q}_h(e^{jh}, khe^{jh})$.

(iii) *Any set $\widetilde{Q}_{rh}(x, y)$ contains at least* $\frac{r(e^{h(\frac{r-5}{2})} - e^{-h(\frac{r-3}{2})})}{e^h - 1} - 2(r + 2)$ *disjoint sets of the form* $\widetilde{Q}_h(e^{jh}, khe^{jh})$.

Proof. (i) Fix any $(x, y) \in \widetilde{\mathbb{A}}$. Then $[\ln x - \frac{h}{2}, \ln x + \frac{h}{2})$ contains a unique integer of the form jh, and for fixed j there exists a unique $a \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$ such that

$\ln x = jh + \ln a$. Further there exists a unique $k \in \mathbb{Z}$ and some $b \in [-\frac{h}{2}, \frac{h}{2})$ such that $y = e^{jh}(kh + b)$. Hence

$$(x, y) = (e^{jh}a, khe^{jh} + e^{jh}b) = (e^{jh}, khe^{jh}) \star (a, b) \in \widetilde{Q}_h(e^{jh}, khe^{jh})$$

with unique $j, k \in \mathbb{Z}$.

(ii) Fix $(x, y) \in \widetilde{\mathbb{A}}$, and suppose that $(u, v) \in \widetilde{Q}_{rh}(x, y) \cap \widetilde{Q}_h(e^{jh}, khe^{jh})$. Then there exist points $(a, b) \in Q_{rh}$, $(c, d) \in Q_h$ such that

$$(u, v) = (x, y) \star (a, b) = (ax, y + xb) \in \widetilde{Q}_{rh}(x, y)$$

and

$$(u, v) = (e^{jh}, khe^{jh}) \star (c, d) = (e^{jh}c, khe^{jh} + e^{jh}d) \in \widetilde{Q}_h(e^{jh}, khe^{jh}).$$

In particular, $\frac{ax}{c} = e^{jh}$ with $e^{-\frac{rh}{2}} \le a < e^{\frac{rh}{2}}$ and $e^{-\frac{h}{2}} \le c < e^{\frac{h}{2}}$, so $\frac{x}{e^{\frac{h}{2}(r+1)}} \le e^{jh} \le xe^{\frac{h}{2}(r+1)}$. Therefore

$$\frac{\ln x}{h} - \frac{r+1}{2} \le j \le \frac{\ln x}{h} + \frac{r+1}{2}.$$

Further, $k = \frac{1}{h}\left(\frac{y}{e^{jh}} + \frac{xb}{e^{jh}} - d\right)$, so

$$\frac{y}{he^{jh}} - \frac{rx}{2e^{jh}} - \frac{1}{2} \le k < \frac{y}{he^{jh}} + \frac{rx}{2e^{jh}} + \frac{1}{2}.$$

For a given value of j, this is satisfied for at most $\frac{rx}{e^{jh}} + 2$ values of k. Thus, $\widetilde{Q}_{rh}(x, y)$ can intersect at most

$$\begin{aligned}
&\sum_{j=\lfloor \frac{\ln x}{h} - \frac{r+1}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{r+1}{2} \rceil} \frac{rx}{e^{jh}} + 2 \\
&\le rx \left[\frac{e^h - e^{-h(\frac{\ln x}{h} + \frac{r+1}{2} + 1)}}{e^h - 1} - \frac{e^h - e^{-h(\frac{\ln x}{h} - \frac{r+1}{2} - 2)}}{e^h - 1} \right] + 2(r+2) \\
&= \frac{r(e^{h(\frac{r+5}{2})} - e^{-h(\frac{r+3}{2})})}{e^h - 1} + 2(r+2)
\end{aligned}$$

sets of the form $\widetilde{Q}_h(e^{jh}, khe^{jh})$.

(iii) The proof is similar to the proof of part (ii). □

Certainly, this fact makes this notion of affine density attractive. However, the following result destroys the nice picture, since it shows that classical affine systems do *not* possess a *uniform* density equal to the ubiquitous constant $\frac{1}{b \ln a}$, such as the density defined in Section 3.1 (see Lemma 3.3). Thus it seems that it is not possible to have both the tiling property and the "correct" uniform affine density for classical affine systems.

Due to the definition of the wavelet transform adapted to the group structure of $\widetilde{\mathbb{A}}$, a classical affine system is now of the form $\widetilde{\mathcal{W}}(\psi, \{(a^j, a^j bk)\}_{j,k \in \mathbb{Z}})$ for $a > 1$, $b > 0$, and $\psi \in L^2(\mathbb{R})$.

Lemma 3.9. *If $\widetilde{\mathcal{W}}(\psi, \Lambda)$ is a classical affine system, then we have*

$$\widetilde{\mathcal{D}}^-(\Lambda) = \frac{1}{b(a-1)} \quad \textit{and} \quad \widetilde{\mathcal{D}}^+(\Lambda) = \frac{a}{b(a-1)}.$$

Proof. Fix $h > 0$ and $(x, y) \in \widetilde{\mathbb{A}}$. If $(a^j, a^j bk) \in \widetilde{Q}_h(x, y)$, then

$$\left(\tfrac{a^j}{x}, \tfrac{a^j bk}{x} - \tfrac{y}{x}\right) = (x, y)^{-1} \star (a^j, a^j bk) \in Q_h.$$

This requires

$$\frac{2 \ln x - h}{2 \ln a} \leq j < \frac{2 \ln x + h}{2 \ln a} \quad \text{and} \quad \frac{2y - xh}{2a^j b} \leq k < \frac{2y + xh}{2a^j b}.$$

Since terms ± 1 are not significant in the limit (cf. the proof of Lemma 3.3), it suffices to observe that

$$\#(\Lambda \cap \widetilde{Q}_h(x, y)) \approx \frac{xh}{b} \sum_{j=\lceil \frac{2\ln x - h}{2 \ln a} \rceil}^{\lceil \frac{2 \ln x + h}{2 \ln a} \rceil - 1} \frac{1}{a^j}.$$

By choosing x appropriately, this implies that

$$\begin{aligned} \widetilde{\mathcal{D}}^-(\Lambda) &= \liminf_{h \to \infty} \inf_{(x,y) \in \widetilde{\mathbb{A}}} \frac{\#(\Lambda \cap \widetilde{Q}_h(x, y))}{h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})} \\ &= \liminf_{h \to \infty} \frac{h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})}{b(a-1)h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})} = \frac{1}{b(a-1)} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathcal{D}}^+(\Lambda) &= \limsup_{h \to \infty} \sup_{(x,y) \in \widetilde{\mathbb{A}}} \frac{\#(\Lambda \cap \widetilde{Q}_h(x, y))}{h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})} \\ &= \limsup_{h \to \infty} \frac{h(ae^{\frac{h}{2}} - e^{-\frac{h}{2}})}{b(a-1)h(e^{\frac{h}{2}} - e^{-\frac{h}{2}})} = \frac{a}{b(a-1)}, \end{aligned}$$

which finishes the proof. □

In [119] Sun and Zhou defined a notion of affine density for nonweighted sequences by employing the affine group $\widetilde{\mathbb{A}}$. However, due to Lemma 3.9, they had to use weights in the computation of both the cardinality $\#(\Lambda \cap \widetilde{Q}_h(x, y))$ and the volume $\mu_{\widetilde{\mathbb{A}}}(\widetilde{Q}_h(x, y))$ to ensure that classical affine systems have a uniform affine density equal to $\frac{1}{b \ln a}$, cf. [119, Rem. 3.1]. In our approach classical affine systems possess this uniform affine density without adding artificial weights. We further remark that Sun and Zhou chose the boxes $\widetilde{Q}_h$ to be dependent on two parameters; thus it is possible to change the height and the width of the boxes independently. But it can be seen that this does not make any difference.

However, we can show that if we choose another exhaustive sequence of boxes in $\widetilde{\mathbb{A}}$, we in fact obtain the "correct" density for classical affine systems without the necessity to add weights. But then we loose the property of having a sequence of rectangular neighborhoods of the identity and the property of tiling the affine group at each level h.

To prove this result we employ the group isomorphism connecting $\mathbb{A}$ and $\widetilde{\mathbb{A}}$.

Remark 3.10. Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ be defined by

$$\Phi(x, y) = (x, xy).$$

We have

$$\Phi((a, b) \cdot (x, y)) = (ax, ab + axy) = \Phi(a, b) \star \Phi(x, y).$$

Since Φ is also bijective, $\Phi : \mathbb{A} \to \widetilde{\mathbb{A}}$ is the wanted group isomorphism.

Now define $(\widetilde{K}_h)_{h>0}$ to be the exhaustive sequence of neighborhoods of the identity in $\widetilde{\mathbb{A}}$ given by $\widetilde{K}_h = \Phi(Q_h)$ and let $\widetilde{K}_h(x, y)$ for $(x, y) \in \widetilde{\mathbb{A}}$ be defined by $\widetilde{K}_h(x, y) = (x, y) \star \widetilde{K}_h$. Then this choice of neighborhoods has the desired property.

Lemma 3.11. *Let $a > 1$, $b > 0$, and define $\Lambda = \{(a^j, a^j bk)\}_{j,k\in\mathbb{Z}}$, i.e., Λ is the sequence in $\widetilde{\mathbb{A}}$ associated with a classical affine system. Then*

$$\limsup_{h\to\infty} \sup_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\#(\Lambda \cap \widetilde{K}_h(x, y))}{\mu_{\widetilde{\mathbb{A}}}(\widetilde{K}_h)} = \liminf_{h\to\infty} \inf_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\#(\Lambda \cap \widetilde{K}_h(x, y))}{\mu_{\widetilde{\mathbb{A}}}(\widetilde{K}_h)} = \frac{1}{b \ln a}.$$

Proof. First notice that

$$\mu_{\widetilde{\mathbb{A}}}(\widetilde{K}_h) = \mu_{\widetilde{\mathbb{A}}}(\Phi(Q_h)) = \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{xh}{2}}^{\frac{xh}{2}} dy \frac{dx}{x^2} = h^2.$$

Since Φ is an isomorphism, we obtain

$$\begin{aligned}
&\limsup_{h\to\infty} \sup_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\#(\Lambda \cap \widetilde{K}_h(x, y))}{\mu_{\widetilde{\mathbb{A}}}(\widetilde{K}_h)} \\
&= \limsup_{h\to\infty} \sup_{(x,y)\in\widetilde{\mathbb{A}}} \frac{\#(\Phi(\Phi^{-1}(\Lambda)) \cap (x, y) \star \Phi(Q_h))}{h^2} \\
&= \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\#(\Phi(\Phi^{-1}(\Lambda) \cap Q_h(x, y)))}{h^2} \\
&= \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{A}} \frac{\#(\Phi^{-1}(\Lambda) \cap Q_h(x, y))}{h^2}
\end{aligned}$$

$$= \mathcal{D}^+(\Phi^{-1}(\Lambda))$$
$$= \frac{1}{b \ln a}.$$

The last equality follows from Lemma 3.3. A similar argument proves the remaining claim. □

3.4 Comparison of Both Notions of Affine Density

In the following we will show that although the two affine densities differ in the exact value for classical affine systems, they are equivalent in a special sense, i.e., they are equal in a *qualitative* sense concerning the upper density.

For this, we need the following two technical results, which are similar to Propositions 3.6 and 3.4. Since the proofs are similar to those (only Lemma 3.8 instead of Lemma 3.5 is used), we omit them.

Proposition 3.12. *If $\Lambda \subseteq \widetilde{\mathbb{A}}$ and $w : \Lambda \to \mathbb{R}^+$, then the following conditions are equivalent.*

(i) $\widetilde{\mathcal{D}}^+(\Lambda, w) < \infty$.
(ii) *There exists $h > 0$ such that* $\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_w(\Lambda \cap \widetilde{Q}_h(x,y)) < \infty$.
(iii) *For every $h > 0$ we have* $\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_w(\Lambda \cap \widetilde{Q}_h(x,y)) < \infty$.

Proposition 3.13. *Let $\Lambda_1, \ldots, \Lambda_L \subseteq \widetilde{\mathbb{A}}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then we have*

$$\widetilde{\mathcal{D}}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty \iff \widetilde{\mathcal{D}}^+(\Lambda_\ell, w_\ell) < \infty \text{ for all } \ell = 1, \ldots, L.$$

The next result states the above-mentioned equivalence in a precise way.

Theorem 3.14. *Let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the following conditions are equivalent.*

(i) $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.
(ii) $\widetilde{\mathcal{D}}^+(\{(\Phi(\Lambda_\ell), w_\ell \circ \Phi^{-1})\}_{\ell=1}^L) < \infty$.

Proof. Notice that by employing Propositions 3.4 and 3.13, it suffices to only consider the case $L = 1$. Hence we let Λ be a sequence in $\mathbb{A}$ with associated weight function given by $w : \Lambda \to \mathbb{R}^+$.

By Proposition 3.6, (i) holds if and only if there exists some $h > 0$ such that we have $\sup_{(x,y)\in\mathbb{A}} \#_w(\Lambda \cap Q_h(x,y)) < \infty$. Now $\#_w(\Lambda \cap Q_h(x,y)) = \#_{w\circ\Phi^{-1}}(\Phi(\Lambda) \cap \Phi(Q_h(x,y)))$ and $\Phi(Q_h(x,y)) = \Phi(x,y) \star \Phi(Q_h)$. This shows that (i) is equivalent to the existence of some $h > 0$ such that

$$\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star\Phi(Q_h)) < \infty.$$

On the other hand, using Proposition 3.12, we get that (ii) holds if and only if there exists $h > 0$ such that $\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_h) < \infty$. It is easy to check that we have $Q_{he^{-\frac{h}{2}}} \subseteq \Phi(Q_h) \subseteq Q_{he^{\frac{h}{2}}}$ for all $h > 0$. This implies

$$\begin{aligned}
&\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_{he^{-\frac{h}{2}}})\\
&\le \sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star\Phi(Q_h))\\
&\le \sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_{he^{\frac{h}{2}}}),
\end{aligned}$$

which shows that there exists $h > 0$ such that

$$\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star\Phi(Q_h)) < \infty$$

if and only if there exists $h > 0$ such that

$$\sup_{(x,y)\in\widetilde{\mathbb{A}}} \#_{w\circ\Phi^{-1}}(\Phi(\Lambda)\cap(x,y)\star Q_h) < \infty.$$

This shows that (i) and (ii) are equivalent. □

In Theorem 4.1 it will be shown that $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty$ is a necessary condition for $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell)$ to possess an upper frame bound. Thus since the relation $\mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell) = \widetilde{\mathcal{W}}(\psi_\ell, \Phi(\Lambda_\ell), w_\ell \circ \Phi^{-1})$ holds, concerning determining necessary conditions on the existence of upper frame bounds for wavelet systems both affine densities yield the same results. However, in Chapter 5 we will show that affine density can also be related to the frame bounds of a wavelet frame and the admissibility constant of the generator in which case the exact value of the affine density will indeed play a vital role.

4

Qualitative Density Conditions

We prove that there exist necessary conditions on a weighted wavelet system with finitely many generators in order that it possesses frame bounds. Specifically, we prove that if such a system possesses an upper frame bound, then the upper weighted affine density is finite. Further, under some hypotheses, we prove that if such a system possesses a lower frame bound, then the lower affine density is strictly positive.

We then apply these results to oversampled affine systems (which include the classical affine and the quasi-affine systems as special cases), to co-affine systems, and to systems consisting only of dilations, obtaining some new results relating density to the frame properties of these systems.

In 2002 Hernández, Labate, and Weiss [77] gave a characterization of when a weighted wavelet system with finitely many generators and with arbitrary sequences of scale indices forms a Parseval frame, assuming that a certain hypothesis known as the local integrability condition (LIC) holds. We show that, under some mild regularity assumption on the analyzing wavelets, the LIC is solely a density condition on the sequences of scale indices, thereby emphasizing the utility of the notion of density. More precisely, it will be proven that the LIC is equivalent to the condition that the weighted sequences of scale indices possess a finite upper weighted density. This condition is very natural, since applying the results obtained in the first part of this chapter shows that wavelet frames of the considered form must have finite upper weighted density.

The main results in this chapter generalize the results obtained in Heil and Kutyniok [73] to multiple generators and the results obtained in Kutyniok [94] to wavelet systems with finitely many generators equipped with weights.

4.1 Existence of an Upper Frame Bound

First we study necessary conditions for the existence of an upper frame bound, i.e., for a wavelet system to be Bessel, in terms of affine density. Intuitively, a wavelet system being Bessel should imply that its sequences of time-scale indices do not contain "too many" elements, i.e., that there cannot be too much "crowding together" of points. And in fact, affine density gives a precise formalism for this intuition. This result is inspired by a similar result for unweighted Gabor systems by Christensen, Deng, and Heil [22, Thm. 1.1(a)]. We remark that a similar result for singly generated nonweighted wavelet systems was simultaneously derived by Sun and Zhou [119, Thm. 3.2] for their notion of density.

Theorem 4.1. *Let $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})\backslash\{0\}$, and let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ be given with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$. If the system $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell)$ possesses an upper frame bound for $L^2(\mathbb{R})$, then $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.*

Proof. Assume that $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ are given such that $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \infty$. We will show that $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell)$ does not possess an upper frame bound for any collection of analyzing wavelets $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R}) \setminus \{0\}$.

Fix some $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$. Since $\mathcal{D}^+(\{(\Lambda_\ell, w_\ell)\}_{\ell=1}^L) = \infty$, by Proposition 3.4(i) there exists $\ell_0 \in \{1, \ldots, L\}$ with $\mathcal{D}^+(\Lambda_{\ell_0}, w_{\ell_0}) = \infty$. Now since $W_{\psi_{\ell_0}} f$ is nonzero and continuous, there must exist some $(c, d) \in \mathbb{A}$ and some $h > 0$ such that $W_{\psi_{\ell_0}} f$ does not vanish on the closure of $Q_h(c, d)$, and consequently,

$$\inf_{(x,y)\in Q_h(c,d)} |W_{\psi_{\ell_0}} f(x, y)| = \delta > 0.$$

Now choose any $N > 0$. Since $\mathcal{D}^+(\Lambda_{\ell_0}, w_{\ell_0}) = \infty$, it follows from Proposition 3.6 that

$$\sup_{(x,y)\in\mathbb{A}} \#_{w_{\ell_0}}(\Lambda_{\ell_0} \cap Q_h(x, y)) = \infty,$$

so there must exist a point $(p, q) \in \mathbb{A}$ such that $\#_{w_{\ell_0}}(\Lambda_{\ell_0} \cap Q_h(p, q)) \geq N$. Define

$$g = \sigma((p, q) \cdot (c, d)^{-1})f$$

and note that $\|g\|_2 = \|f\|_2 = 1$. Now,

$$(a, b) \in Q_h(p, q) = (p, q)\cdot Q_h \implies (c, d)\cdot(p, q)^{-1}\cdot(a, b) \in (c, d)\cdot Q_h = Q_h(c, d),$$

so we can compute that

$$\begin{aligned}
&\sum_{(a,b)\in\Lambda_{\ell_0}} |\langle g, w_{\ell_0}(a, b)^{\frac{1}{2}}\, \sigma(a, b)\psi_{\ell_0}\rangle|^2 \\
&\geq \sum_{(a,b)\in\Lambda_{\ell_0}\cap Q_h(p,q)} |\langle\sigma((p, q)\cdot(c, d)^{-1})f, w_{\ell_0}(a, b)^{\frac{1}{2}}\sigma(a, b)\psi_{\ell_0}\rangle|^2 \\
&= \sum_{(a,b)\in\Lambda_{\ell_0}\cap Q_h(p,q)} |w_{\ell_0}(a, b)|\, \left|\left\langle f, \sigma((c, d)\cdot(p, q)^{-1}\cdot(a, b))\psi_{\ell_0}\right\rangle\right|^2
\end{aligned}$$

$$= \sum_{(a,b)\in\Lambda_{\ell_0}\cap Q_h(p,q)} |w_{\ell_0}(a,b)|\,|W_{\psi_{\ell_0}} f((c,d)\cdot(p,q)^{-1}\cdot(a,b))|^2$$
$$\geq \delta^2\,\#_{w_{\ell_0}}(\Lambda_{\ell_0}\cap Q_h(p,q))$$
$$\geq \delta^2 N.$$

Since N is arbitrary and $\|g\|_2 = 1$, we conclude that $\mathcal{W}(\psi_{\ell_0}, \Lambda_{\ell_0}, w_{\ell_0})$, and hence in particular $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell)$, cannot possess an upper frame bound. □

4.2 Existence of a Lower Frame Bound

In a similar way as for the upper frame bound, our intuition tells us that the existence of a lower frame bound for a wavelet system should imply that the associated sequences of time-scale indices should possess "enough" elements, i.e., there should not be "gaps" in the distribution of points of the sequences of time-scale indices of arbitrarily large size. Due to Theorem 4.1, we expect that, provided a weighted wavelet system with finitely many (admissible) generators possesses a lower frame bound, the lower weighted affine density of its weighted sequences of time-scale indices should be at least positive. This general claim and even the claim for unweighted wavelet systems is still unsolved. In this section we will prove the following result which does not restrict the choice of the analyzing wavelets, but only poses restrictions on the sequences of time-scale indices. In Section 6.4 we will derive another partial result (Corollary 6.12) for *any* finite number of sequences of time-scale indices but with a mild regularity condition on the analyzing wavelets. We also refer to partial results derived by Sun in [117].

Theorem 4.2. *Let $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$, and let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with disjoint union denoted by $\Lambda = \bigcup_{\ell=1}^{L} \Lambda_\ell$. Suppose that $\mathcal{D}^+(\bigcup_{\ell=1}^{L} \Lambda_\ell^{-1}) < \infty$. If $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ possesses a lower frame bound for $L^2(\mathbb{R})$, then $\mathcal{D}^-(\Lambda) > 0$.*

Before stating the proof of Theorem 4.2, in addition to Proposition 3.6, we give a further interpretation of finite upper affine density in terms of the following definition.

Definition 4.3. *We will say that a set $K \subseteq \mathbb{A}$ is* affinely h-separated *if*

$$(a,b) \neq (c,d) \in K \implies Q_h(a,b) \cap Q_h(c,d) = \emptyset.$$

Before proving a characterization of finite upper affine density in terms of this notion, we require the following technical lemma.

Lemma 4.4. *Let $h > 0$ be given. If $Q_h(x,y) \cap Q_h(a,b) \neq \emptyset$, then $(x,y) \in Q_{2he^{\frac{h}{2}}}(a,b)$.*

Proof. Suppose that $(c,d) \in Q_h(x,y) \cap Q_h(a,b)$. Then we would have

$$(c,d) = (x,y)(t,u) = (a,b)(r,s)$$

for some (t,u), $(r,s) \in Q_h$. Therefore,

$$(a,b)^{-1} \cdot (x,y) = (r,s) \cdot (t,u)^{-1} = (\tfrac{r}{t}, st - tu) \in Q_{2he^{\frac{h}{2}}},$$

so $(x,y) \in Q_{2he^{\frac{h}{2}}}(a,b)$. □

Now we derive a further interpretation of finite upper affine density. This should be compared to the characterization result in Proposition 3.6.

Proposition 4.5. *If $\Lambda \subseteq \mathbb{A}$, then the following conditions are equivalent.*

(i) $\mathcal{D}^+(\Lambda) < \infty$.

(ii) *There exists $h > 0$ such that Λ can be written as a finite union of sequences $\Lambda_1, \ldots, \Lambda_N$, each of which is affinely h-separated.*

(iii) *For every $h > 0$, Λ can be written as a finite union of sequences $\Lambda_1, \ldots, \Lambda_N$, each of which is affinely h-separated.*

Proof. (i) $\Rightarrow$ (iii). Assume that $D^+(\Lambda) < \infty$, and let $h > 0$ be given. Then by Proposition 3.6, we have $M = \sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap Q_h(x,y)) < \infty$. Fix any $(a,b) \in \Lambda$. If $(c,d) \in \Lambda$ is such that $Q_h(a,b) \cap Q_h(c,d) \neq \emptyset$, then we have by Lemma 4.4 that $(c,d) \in Q_{2he^{\frac{h}{2}}}(a,b)$. Now by Lemma 3.5(ii), there exists an integer N, independent of (a,b), such that $Q_{2he^{\frac{h}{2}}}(a,b)$ is contained in a union of at most N sets of the form $Q_h(e^{jh}, khe^{-\frac{h}{2}})$. However, each set $Q_h(e^{jh}, khe^{-\frac{h}{2}})$ can contain at most M points of Λ. Hence $Q_{2he^{\frac{h}{2}}}(a,b)$ can contain at most MN points of Λ.

Thus, each $Q_h(a,b)$ with $(a,b) \in \Lambda$ can intersect at most MN sets $Q_h(c,d)$ with $(c,d) \in \Lambda$. By the disjointization principle of Feichtinger and Gröbner [51, Lem. 2.9], it follows that Λ can be divided into at most MN subsequences $\Lambda_1, \ldots, \Lambda_{MN}$ such that for each fixed i, the sets $Q_h(a,b)$ with $(a,b) \in \Lambda_i$ are disjoint, or in other words, Λ_i is affinely h-separated.

(ii) $\Rightarrow$ (i). Assume that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ with each Λ_i affinely h-separated. Fix δ so that $1 < 2\delta e^{\frac{\delta}{2}} < h$, and suppose that two points (a,b) and (c,d) of some Λ_i were both contained in some $Q_\delta(x,y)$. Then by Lemma 4.4, we would have $(x,y) \in Q_{2\delta e^{\frac{\delta}{2}}}(a,b) \subseteq Q_h(a,b)$ and $(x,y) \in Q_{2\delta e^{\frac{\delta}{2}}}(c,d) \subseteq Q_h(c,d)$. Hence $(a,b) = (c,d)$ since Λ_i is affinely h-separated. Thus, each $Q_\delta(x,y)$ contains at most one point of Λ_i, so $\sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap Q_\delta(x,y)) \le N < \infty$. It therefore follows from Proposition 3.6 that $\mathcal{D}^+(\Lambda) < \infty$.

Since (iii) $\Rightarrow$ (ii) is obvious, the proposition is proved. □

For the proof of our main result in this section, in addition, we require the following technical lemmas.

Lemma 4.6. *If $\Lambda \subseteq \mathbb{A}$ satisfies $\mathcal{D}^+(\Lambda) < \infty$, then $\mathcal{D}^+(\Lambda \cdot (p,q)) < \infty$ for each $(p,q) \in \mathbb{A}$.*

Proof. Since $\mathcal{D}^+(\Lambda) < \infty$, we have by Proposition 3.6 that

$$M = \sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap Q_1(x,y)) < \infty.$$

Fix any $(p,q) \in \mathbb{A}$. By Lemma 3.5, we have that $\{Q_1(e^j, ke^{-\frac{1}{2}}) \cdot (p,q)\}_{j,k\in\mathbb{Z}}$ covers $\mathbb{A}$, and there exists an integer N independent of (x,y) such that each $Q_1(x,y)$ intersects at most N of the sets $Q_1(e^j, ke^{-\frac{1}{2}}) \cdot (p,q)$. Therefore,

$$\begin{aligned}\#(\Lambda \cdot (p,q) \cap Q_1(x,y)) &\le N \sup_{j,k\in\mathbb{Z}} \#(\Lambda \cdot (p,q) \cap Q_1(e^j, ke^{-\frac{1}{2}}) \cdot (p,q)) \\ &= N \sup_{j,k\in\mathbb{Z}} \#(\Lambda \cap Q_1(e^j, ke^{-\frac{1}{2}})) \\ &\le NM,\end{aligned}$$

and therefore $\mathcal{D}^+(\Lambda \cdot (p,q)) < \infty$ by Proposition 3.6. □

Lemma 4.7. *Let $\delta, R > 1$ be given. If $T > e^{\frac{R}{2}}(R+\delta)$, then we have for every $(p,q) \in \mathbb{A}$ that*

$$(a,b) \notin Q_T(p,q) \implies Q_R \cap Q_\delta((a,b)^{-1} \cdot (p,q)) = \emptyset.$$

Proof. Suppose that there exists a point $(x,y) \in Q_R \cap Q_\delta((a,b)^{-1} \cdot (p,q))$. Then $(x,y) = (a,b)^{-1} \cdot (p,q) \cdot (c,d)$ for some $(c,d) \in Q_\delta$. Since we also have $(x,y) \in Q_R$, we can check that

$$(p,q)^{-1} \cdot (a,b) = (c,d) \cdot (x,y)^{-1} = (\tfrac{c}{x}, dx - xy) \in Q_T.$$

Therefore $(a,b) \in (p,q) \cdot Q_T = Q_T(p,q)$. □

The following lemma is similar to a result by Christensen, Deng, and Heil [22, Lem. 3.3] for Gabor systems, where here we make use of the Bergman transform instead of the Bargmann transform and choose a different function η.

Lemma 4.8. *Let $\psi \in L^2(\mathbb{R})$, and define $\eta \in L^2_A(\mathbb{R})$ by*

$$\hat{\eta}(\xi) = \begin{cases} 2\xi\, e^{-\xi}, & \xi \ge 0, \\ 0, & \xi < 0. \end{cases}$$

For each $\delta > 1$, there exists a constant $C_\delta > 0$ such that for every $(p,q), (a,b) \in \mathbb{A}$,

$$|\langle \sigma(p,q)\eta, \sigma(a,b)\psi\rangle|^2 \le C_\delta \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} \big|\langle \psi,\, \sigma(x,y)\eta\rangle\big|^2 \, d\mu_{\mathbb{A}}(x,y).$$

Proof. It is easy to check that $\|\eta\|_2 = 1$ and $C_\eta = 1 < \infty$. Therefore η is an admissible function. The wavelet transform $\mathcal{W}_\eta f$ associated with this particular analyzing wavelet η possesses a stronger property than continuity. Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the complex upper half-plane. The *Bergman transform* of $f \in L^2(\mathbb{R})$ is the function Gf defined on $\mathbb{C}^+$ by

$$Gf(b+ai) = \frac{1}{2\pi a^{\frac{3}{2}}} \langle f, \tfrac{1}{\sqrt{a}}\eta(\tfrac{\cdot - b}{a})\rangle = \frac{1}{2\pi a^{\frac{3}{2}}} \left\langle f, \sigma(a, \tfrac{b}{a})\eta \right\rangle = \frac{1}{2\pi a^{\frac{3}{2}}} \mathcal{W}_\eta f(a, \tfrac{b}{a}).$$

For each $f \in L^2(\mathbb{R})$, Gf is an analytic function on $\mathbb{C}^+$, cf. [41, Sec. 2.5], [68, p. 308]. In particular, if we identify $\mathbb{A}$ with $\mathbb{C}^+$ in the obvious way, then since $\overline{Q}_h$ is a compact neighborhood of $i = 0+1i$ in $\mathbb{C}^+$, we have by [84, Thm. 2.2.3] that for each $h > 0$ there exists a constant K_h, independent of $f \in L^2(\mathbb{R})$, such that

$$|Gf(i)| = |Gf(0+1i)| \le K_h \iint_{Q_h} |Gf(z)|\, dz. \tag{4.1}$$

Now let $\delta > 1$ and $(p,q), (a,b) \in \mathbb{A}$ be given. Set $f = \sigma((p,q)^{-1} \cdot (a,b))\psi$ and choose $h > 0$ such that $he^{\frac{h}{2}} = \delta$. Using (4.1), we compute as follows:

$$\begin{aligned}
&|\langle \sigma(p,q)\eta, \sigma(a,b)\psi\rangle|^2 \\
&= \left|\left\langle \eta, \sigma((p,q)^{-1}\cdot(a,b))\psi \right\rangle\right|^2 \\
&= |\langle f, \sigma(1,0)\eta\rangle|^2 \\
&= (2\pi)^2\, |Gf(i)|^2 \\
&\le 4\pi^2 \left(K_h \iint_{Q_h} |Gf(z)|\, dz \right)^2 \\
&= 4\pi^2 K_h^2 \left(\iint_{Q_h} \frac{1}{2\pi x^{\frac{3}{2}}} \left|\left\langle f, \sigma(x, \tfrac{y}{x})\eta \right\rangle\right| \, dy\, dx \right)^2 \\
&= K_h^2 \left(\int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{x^{\frac{3}{2}}} \left|\left\langle f, \sigma(x, \tfrac{y}{x})\eta \right\rangle\right| \, dy\, dx \right)^2 \\
&= K_h^2 \left(\int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{h}{2x}}^{\frac{h}{2x}} \frac{1}{x^{\frac{3}{2}}} |\langle f, \sigma(x,t)\eta\rangle|\, x\, dt\, dx \right)^2 \\
&\le K_h^2 \left(\int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{h}{2}e^{\frac{h}{2}}}^{\frac{h}{2}e^{\frac{h}{2}}} \frac{1}{x^{\frac{1}{2}}} |\langle f, \sigma(x,t)\eta\rangle|\, dt\, dx \right)^2 \\
&\le K_h^2 \left(\int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{h}{2}e^{\frac{h}{2}}}^{\frac{h}{2}e^{\frac{h}{2}}} dt\, dx \right) \left(\iint_{Q_{he^{\frac{h}{2}}}} |\langle f, \sigma(x,t)\eta\rangle|^2\, dt\, \frac{dx}{x} \right) \\
&= C_\delta \iint_{Q_\delta} |\langle f, \sigma(x,t)\eta\rangle|^2\, d\mu_{\mathbb{A}}(x,t)
\end{aligned}$$

$$= C_\delta \iint_{Q_\delta} \left|\left\langle \sigma((p,q)^{-1} \cdot (a,b))\psi, \sigma(x,t)\eta \right\rangle\right|^2 \, d\mu_{\mathbb{A}}(x,t)$$
$$= C_\delta \iint_{Q_\delta} \left|\left\langle \psi, \sigma((a,b)^{-1} \cdot (p,q) \cdot (x,t))\eta \right\rangle\right|^2 \, d\mu_{\mathbb{A}}(x,t)$$
$$= C_\delta \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} \left|\langle \psi, \sigma(x,t)\eta \rangle\right|^2 \, d\mu_{\mathbb{A}}(x,t).$$

This proves the claim. □

Now we can give the proof of Theorem 4.2.

Proof (of Theorem 4.2). Assume that functions $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ and sequences $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ are given such that $\mathcal{D}^+(\bigcup_{\ell=1}^L \Lambda_\ell^{-1}) < \infty$ and $\mathcal{D}^-(\bigcup_{\ell=1}^L \Lambda_\ell) = 0$. Our goal is to show that $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell)$ does not possess a lower frame bound.

Employing Proposition 3.4(i) we obtain that we have $\mathcal{D}^+(\Lambda_\ell^{-1}) < \infty$ for all $\ell = 1, \dots, L$. Now fix $\varepsilon > 0$, and let $\eta \in L^2(\mathbb{R})$ denote the analyzing wavelet defined in Lemma 4.8. We have $W_\eta \psi_\ell \in L^2(\mathbb{A}, d\mu_{\mathbb{A}})$, so, since the sets Q_h, $h > 0$ form a nested, increasing, exhaustive sequence of subsets of $\mathbb{A}$ there must exist some $R > 1$ such that

$$\iint_{\mathbb{A}\backslash Q_R} \left|\left\langle \psi_\ell, \, \sigma(x,y)\eta \right\rangle\right|^2 d\mu_{\mathbb{A}}(x,y) < \varepsilon \quad \text{for all } \ell = 1, \dots, L. \tag{4.2}$$

Fix $T > e^{\frac{R}{2}}(R+2)$. Then, since $\mathcal{D}^-(\bigcup_{\ell=1}^L \Lambda_\ell) = 0$, by Proposition 3.7 and the definition of disjoint union we can find a point $(p,q) \in \mathbb{A}$ such that

$$\Lambda_\ell \cap Q_T(p,q) = \emptyset \quad \text{for all } \ell = 1, \dots, L.$$

Let $\delta = 2$. Then we have $e^{\frac{R}{2}}(R+\delta) < T$, and, by Lemma 4.7, we obtain

$$\bigcup_{(a,b)\in\Lambda_\ell} Q_\delta((a,b)^{-1} \cdot (p,q)) \subseteq \mathbb{A}\backslash Q_R \quad \text{for all } \ell = 1, \dots, L.$$

Now, since $\mathcal{D}^+(\Lambda_\ell^{-1}) < \infty$, we have by Lemma 4.6 that $\mathcal{D}^+(\Lambda_\ell^{-1} \cdot (p,q)) < \infty$ as well. Therefore, by Proposition 4.5 we can write each Λ_ℓ as a finite union $\Lambda_\ell = \Lambda_{\ell 1} \cup \dots \cup \Lambda_{\ell N_\ell}$ in such a way that $\Lambda_{\ell i}^{-1} \cdot (p,q)$ is affinely δ-separated for each $i = 1, \dots, N_\ell$. Consequently, for each ℓ and i we have

$$\bigcup_{(a,b)\in\Lambda_{\ell i}} Q_\delta((a,b)^{-1} \cdot (p,q)) \subseteq \mathbb{A}\backslash Q_R, \tag{4.3}$$

with the union being disjoint.

Finally, applying Lemma 4.8 and equations (4.2) and (4.3), we compute that

$$\begin{aligned}
&\sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda_\ell} |\langle \sigma(p,q)\eta, \sigma(a,b)\psi_\ell\rangle|^2 \\
&\le C_\delta \sum_{\ell=1}^{L}\sum_{i=1}^{N_\ell} \sum_{(a,b)\in\Lambda_{\ell i}} \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} |\langle \psi_\ell, \sigma(x,y)\eta\rangle|^2 \, d\mu_{\mathbb{A}}(x,y) \\
&\le C_\delta \sum_{\ell=1}^{L}\sum_{i=1}^{N_\ell} \iint_{\mathbb{A}\setminus Q_R} |\langle \psi_\ell, \sigma(x,y)\eta\rangle|^2 \, d\mu_{\mathbb{A}}(x,y) \\
&\le C_\delta \, L \max_{\ell=1,\dots,L} \{N_\ell\}\, \varepsilon.
\end{aligned}$$

Since $\|\sigma(p,q)\eta\|_2 = \|\eta\|_2 = 1$, it follows that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ cannot possess a lower frame bound, which completes the proof. □

Even in the situation of a single generator, we do not know if the hypothesis $\mathcal{D}^+(\Lambda^{-1}) < \infty$ in Theorem 4.2 is necessary. Adding the assumption $\mathcal{D}^+(\Lambda) < \infty$ does not resolve this question, since, for example, if Λ is the sequence associated with the classical affine system, then Λ^{-1} is the sequence corresponding to the co-affine system, so for this Λ we have $\mathcal{D}^+(\Lambda) < \infty$ yet $\mathcal{D}^+(\Lambda^{-1}) = \infty$. Further, it is not true that if both $\mathcal{D}^+(\Lambda) < \infty$ and $\mathcal{D}^+(\Lambda^{-1}) < \infty$ then necessarily $\mathcal{D}^-(\Lambda) > 0$; for example, consider the sequence $\Lambda = \{(a^j, bk)\}_{j\in\mathbb{Z},k\ge 0}$.

4.3 Examples of Wavelet Systems

In this section we will apply our results derived in the previous two sections to several types of weighted wavelet systems.

We saw already that the sequence of time-scale indices of classical affine systems has uniform affine density equal to the constant $\frac{1}{b\ln a}$ (Lemma 3.3). This is a special case of the following result which proves this property for any oversampled affine system.

Proposition 4.9. *Let $a > 1$ and $b > 0$, and define $\Lambda \subseteq \mathbb{A}$ and $w : \Lambda \to \mathbb{R}^+$ by $\Lambda = \big\{(a^j, \frac{bk}{r_j})\big\}_{j,k\in\mathbb{Z}}$ and $w(a^j, \frac{bk}{r_j}) = \frac{1}{r_j}$. Then (Λ, w) has uniform weighted affine density*

$$\mathcal{D}(\Lambda, w) = \frac{1}{b\ln a}.$$

Proof. Fix $(x,y) \in \mathbb{A}$. If $(a^j, \frac{bk}{r_j}) \in Q_h(x,y)$, then

$$\Big(\tfrac{a^j}{x}, \tfrac{bk}{r_j} - \tfrac{xy}{a^j}\Big) = (x,y)^{-1}\cdot(a^j, \tfrac{bk}{r_j}) \in Q_h.$$

In particular, $\frac{a^j}{x} \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$. There are at least $\frac{h}{\ln a}$ and at most $\frac{h}{\ln a} + 1$ integers j satisfying this condition. Additionally, we have $\frac{xyr_j}{a^jb} - \frac{hr_j}{2b} \le k <$

$\frac{xyr_j}{a^j b} + \frac{hr_j}{2b}$. For a given j, there are at least $\frac{hr_j}{b}$ and at most $\frac{hr_j}{b} + 1$ integers k satisfying this condition. Taking the weight into account, we conclude that

$$\frac{h}{\ln a} \cdot \frac{1}{r_j} \cdot \frac{hr_j}{b} \le \#_w(\Lambda \cap Q_h(x,y)) \le \left(\frac{h}{\ln a} + 1\right) \frac{1}{r_j} \left(\frac{hr_j}{b} + 1\right).$$

Thus

$$\mathcal{D}^+(\Lambda, w) \le \limsup_{h\to\infty} \frac{(\frac{h}{\ln a} + 1)\frac{1}{r_j}(\frac{hr_j}{b} + 1)}{h^2} = \frac{1}{b \ln a},$$

and similarly $\mathcal{D}^-(\Lambda, w) \ge \frac{1}{b \ln a}$. □

Next we will consider the co-affine systems studied in Gressman, Labate, Weiss, and Wilson [61]. By employing merely density conditions, we show that unweighted co-affine systems can possess neither an upper nor a lower frame bound, thereby rediscovering a result in [61] for these systems in the case $b = 1$.

Proposition 4.10. *Let $a > 1$ and $b > 0$, and define $\Lambda \subseteq \mathbb{A}$ by $\Lambda = \{(a^j, \frac{bk}{a^j})\}_{j,k\in\mathbb{Z}}$. Then*

$$\mathcal{D}^-(\Lambda) = 0 \qquad \text{and} \qquad \mathcal{D}^+(\Lambda) = \infty.$$

Consequently, for any $\psi \in L^2(\mathbb{R})$, $\psi \neq 0$, a co-affine system $\mathcal{W}(\psi, \Lambda)$ cannot possess an upper or a lower frame bound.

Proof. Fix $(x, y) \in \mathbb{A}$. If $(a^j, \frac{bk}{a^j}) \in Q_h(x, y)$, then

$$\left(\tfrac{a^j}{x}, \tfrac{bk}{a^j} - \tfrac{xy}{a^j}\right) = (x, y)^{-1} \cdot (a^j, \tfrac{bk}{a^j}) \in Q_h.$$

This requires

$$\frac{2\ln x - h}{2 \ln a} \le j < \frac{2 \ln x + h}{2 \ln a}$$

and

$$\frac{2xy - a^j h}{2b} \le k < \frac{2xy + a^j h}{2b}.$$

As in the proof of Proposition 4.9, terms ± 1 are not significant in the limit, so it suffices to observe that

$$\#(\Lambda \cap Q_h(x, y)) \approx \sum_{j=\lceil \frac{2\ln x - h}{2\ln a} \rceil}^{\lceil \frac{2\ln x + h}{2\ln a} \rceil - 1} \frac{ha^j}{b}.$$

By changing x, we can make this quantity arbitrarily large or small, which yields the conclusion $\mathcal{D}^-(\Lambda) = 0$ and $\mathcal{D}^+(\Lambda) = \infty$. Finally, since Λ^{-1} is the sequence corresponding to the classical affine system, by Lemma 3.3 we have $\mathcal{D}^+(\Lambda^{-1}) = 1/(b \ln a) < \infty$. The nonexistence of frame bounds therefore follows from Theorems 4.1 and 4.2. □

In Section 5.7 we further show that special *weighted* co-affine systems do not exist by again using the notion of affine density.

Finally, we examine systems which consist only of translates or of dilations of a finite set of functions.

Proposition 4.11. *Let* $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$*, let* $T_1, \ldots, T_L \subseteq \mathbb{R}$*, and let* $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ *be given.*

(i) $\{\psi_\ell(x-t)\}_{t\in T_\ell, \ell=1,\ldots,L}$ *is not a frame for* $L^2(\mathbb{R})$*.*

(ii) $\{\frac{1}{\sqrt{s}}\psi_\ell(\frac{x}{s})\}_{s\in S_\ell, \ell=1,\ldots,L}$ *is not a frame for* $L^2(\mathbb{R})$*.*

Proof. Note that both of these systems are special cases of irregular wavelet systems, namely,

$$\{\psi_\ell(x-t)\}_{t\in T_\ell, \ell=1,\ldots,L} = \bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \{1\} \times T_\ell)$$

and

$$\{\tfrac{1}{\sqrt{s}}\psi_\ell(\tfrac{x}{s})\}_{s\in S_\ell, \ell=1,\ldots,L} = \bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times \{0\}).$$

Consider first the case of pure dilations. Note that $\mathcal{D}^-(\bigcup_{\ell=1}^L S_\ell \times \{0\}) = 0$. If $\mathcal{D}^+(\{S_\ell \times \{0\}\}_{\ell=1}^L) = \mathcal{D}^+(\bigcup_{\ell=1}^L S_\ell \times \{0\}) = \infty$ (see Remark 3.2(a) for the first equality), then $\bigcup_{\ell=1}^L \mathcal{D}(\psi_\ell, S_\ell)$ cannot possess an upper frame bound by Theorem 4.1.

Suppose on the other hand that $\mathcal{D}^+(\bigcup_{\ell=1}^L S_\ell \times \{0\}) < \infty$. By Proposition 3.4(i), $\mathcal{D}^+(S_\ell \times \{0\}) < \infty$ for $\ell = 1, \ldots, L$. Fix $\ell \in \{1, \ldots, L\}$. If $(c, 0) \in (S_\ell \times \{0\})^{-1} \cap Q_h(x, y)$, then $(\frac{c}{x}, -\frac{xy}{c}) = (x, y)^{-1} \cdot (c, 0) \in Q_h$. Hence $e^{-\frac{h}{2}} \le \frac{c}{x} < e^{\frac{h}{2}}$, so $-\frac{h}{2}e^{-h} < -\frac{cy}{x} = -\frac{xy}{c}\frac{c}{x}\frac{c}{x} < \frac{h}{2}e^h$. Therefore $(\frac{x}{c}, -\frac{cy}{x}) = (\frac{1}{x}, y)^{-1} \cdot (\frac{1}{c}, 0) \in Q_{he^h}$, so $(\frac{1}{c}, 0) \in Q_{he^h}(\frac{1}{x}, y)$. Thus

$$\sup_{(x,y)\in\mathbb{A}} \#((S_\ell \times \{0\})^{-1} \cap Q_h(x,y)) \le \sup_{(x,y)\in\mathbb{A}} \#((S_\ell \times \{0\}) \cap Q_{he^h}(x,y)) < \infty,$$

so $\mathcal{D}^+((S_\ell \times \{0\})^{-1}) < \infty$ for all $\ell = 1, \ldots, L$ by Proposition 3.6. Thus $\mathcal{D}^+(\bigcup_{\ell=1}^L (S_\ell \times \{0\})^{-1}) < \infty$ by Proposition 3.4(i). Consequently, Theorem 4.2 implies that the system $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times \{0\})$ cannot possess a lower frame bound in this case.

The proof for $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \{1\} \times T_\ell)$ is similar, and was also obtained by Christensen, Deng, and Heil [22] by using the fact that a system of pure translations is a Gabor system of the form $\bigcup_{\ell=1}^L \mathcal{G}(g_\ell, T_\ell \times \{0\})$. □

4.4 Density of Sequences in $\mathbb{R}^+$

In this section we will derive a notion of weighted density for multiple weighted sequences in $\mathbb{R}^+$ adapted to the geometry of the multiplicative group $\mathbb{R}^+$ in the spirit of the definition of Beurling density on Euclidean space and affine density on the affine group. This notion will be employed in the next section as well as also in Chapter 5. Notice that throughout, although S will always denote a sequence of points in $\mathbb{R}^+$ and not merely a subset, for simplicity we will write $S \subseteq \mathbb{R}^+$.

We consider a fixed increasing family of neighborhoods of the identity element 1 in $\mathbb{R}^+$. For simplicity of computation, we will take

$$\{[e^{-\frac{h}{2}}, e^{\frac{h}{2}})\}_{h>0}.$$

Let $\mu_{\mathbb{R}^+} = \frac{dx}{x}$ denote the Haar measure on $\mathbb{R}^+$. Since $\mu_{\mathbb{R}^+}$ is invariant under multiplication, we have that

$$\mu_{\mathbb{R}^+}(x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) = \mu_{\mathbb{R}^+}([e^{-\frac{h}{2}}, e^{\frac{h}{2}})) = \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{dx}{x} = h.$$

Then the upper and lower densities of some multiple weighted sequence $\{(S_\ell, w_\ell)\}_{\ell=1}^L$ in $\mathbb{R}^+$ are defined as follows. Notice that we use the same notion as for the density of sequences in $\mathbb{A}$. The type of density is then always completely determined by the sequence to which it is applied.

Definition 4.12. *Given $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$. Then the* upper weighted density *of $\{(S_\ell, w_\ell)\}_{\ell=1}^L$ is defined by*

$$\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^+} \frac{\sum_{\ell=1}^L \#_{w_\ell}(S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))}{h},$$

and the lower weighted density *of $\{(S_\ell, w_\ell)\}_{\ell=1}^L$ is*

$$\mathcal{D}^-(\{(S_\ell, w_\ell)\}_{\ell=1}^L) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^+} \frac{\sum_{\ell=1}^L \#_{w_\ell}(S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))}{h}.$$

If $\mathcal{D}^-(\{(S_\ell, w_\ell)\}_{\ell=1}^L) = \mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L)$, then $\{(S_\ell, w_\ell)\}_{\ell=1}^L$ has uniform density*, which is denoted by $\mathcal{D}(\{(S_\ell, w_\ell)\}_{\ell=1}^L)$.*

We first make the following basic observations.

Remark 4.13. (a) Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ be given. Then, using the disjoint union $S = \bigcup_{\ell=1}^L S_\ell$ of the sequences $S_1, \ldots, S_L$, we obtain the following simpler form:

$$\mathcal{D}^+(\{S_\ell\}_{\ell=1}^L) = \mathcal{D}^+(S) \quad \text{and} \quad \mathcal{D}^-(\{S_\ell\}_{\ell=1}^L) = \mathcal{D}^-(S).$$

(b) Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then

$$\sum_{\ell=1}^{L} \mathcal{D}^-(S_\ell, w_\ell) \le \mathcal{D}^-(\{(S_\ell, w_\ell)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) \le \sum_{\ell=1}^{L} \mathcal{D}^+(S_\ell, w_\ell).$$

The following result shows that this density is robust against perturbations of the sequence of scale indices.

Lemma 4.14. *Let $S \subseteq \mathbb{R}^+$, $w : S \to \mathbb{R}^+$, and $\varepsilon > 0$. For each $\tilde{S} = \{s\delta_s : s \in S,\ \delta_s \in [e^{-\frac{\varepsilon}{2}}, e^{\frac{\varepsilon}{2}}]\}$ equipped with a weight function $v : \tilde{S} \to \mathbb{R}^+$ defined by $v(s\delta_s) = w(s)$, we have $\mathcal{D}^-(S, w) = \mathcal{D}^-(\tilde{S}, v)$ and $\mathcal{D}^+(S, w) = \mathcal{D}^+(\tilde{S}, v)$.*

Proof. For $h > 0$ and $x \in \mathbb{R}^+$, we obtain the following estimates for $\#_w(S \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))$:

$$\#_v(\tilde{S} \cap x[e^{-\frac{h-\varepsilon}{2}}, e^{\frac{h-\varepsilon}{2}})) \le \#_w(S \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) \le \#_v(\tilde{S} \cap x[e^{-\frac{h+\varepsilon}{2}}, e^{\frac{h+\varepsilon}{2}})).$$

Dividing the terms by h, and observing that

$$\begin{aligned} &\limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^+} \#_v(\tilde{S} \cap x[e^{-\frac{h-\varepsilon}{2}}, e^{\frac{h-\varepsilon}{2}}))}{h} \\ &= \mathcal{D}^+(\tilde{S}, v) \\ &= \limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^+} \#_v(\tilde{S} \cap x[e^{-\frac{h+\varepsilon}{2}}, e^{\frac{h+\varepsilon}{2}}))}{h}, \end{aligned}$$

proves $\mathcal{D}^+(S, w) = \mathcal{D}^+(\tilde{S}, v)$.

The claim concerning the lower density can be treated similarly. □

Next we obtain a useful reinterpretation of finite upper density of a single weighted sequence (compare also Proposition 3.6).

Proposition 4.15. *Let $S \subseteq \mathbb{R}^+$ with weight function $w : S \to \mathbb{R}^+$ be given. Then the following conditions are equivalent.*

(i) $\mathcal{D}^+(S, w) < \infty$.

(ii) *There exists an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$ such that $\sup_{x\in\mathbb{R}^+} \#_w(S \cap xI) < \infty$.*

(iii) *For every interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$, we have $\sup_{x\in\mathbb{R}^+} \#_w(S \cap xI) < \infty$.*

Proof. (i) ⇒ (ii) and (iii) ⇒ (ii) are trivial.

(ii) ⇒ (i), (iii). Suppose there exists an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$ and some constant $N < \infty$ with $\#_w(S \cap xI) < N$ for all $x \in \mathbb{R}^+$. Let J be another interval in $\mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(J) < \infty$. If there exists $y \in \mathbb{R}^+$ with

$yJ \subseteq I$, then $\#_w(S \cap xJ) < N$ for all $x \in \mathbb{R}^+$. On the other hand, if there exists $y \in \mathbb{R}^+$ with $yI \subseteq J$, then $\mu_{\mathbb{R}^+}(J) = r\mu_{\mathbb{R}^+}(I)$ for some $r \geq 1$, and J is covered by a union of at most $r+1$ sets of the form xI. Consequently,

$$\sup_{x \in \mathbb{R}^+} \#_w(S \cap xJ) \leq (r+1) \sup_{x \in \mathbb{R}^+} \#_w(S \cap xI) \leq (r+1)N.$$

Thus statement (iii) holds. Further,

$$\mathcal{D}^+(S) \leq \limsup_{r \to \infty} \frac{\sup_{x \in \mathbb{R}^+} \#_w(S \cap xJ)}{r\mu_{\mathbb{R}^+}(I)} \leq \limsup_{r \to \infty} \frac{(r+1)N}{r\mu_{\mathbb{R}^+}(I)} = \frac{N}{\mu_{\mathbb{R}^+}(I)} < \infty,$$

so statement (i) holds as well. □

The following result for the lower density follows in a similar way.

Proposition 4.16. *Let $S \subseteq \mathbb{R}^+$ with weight function $w : S \to \mathbb{R}^+$ be given. Then the following conditions are equivalent.*

(i) $\mathcal{D}^-(S, w) > 0$.

(ii) *There exists an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$ such that $\inf_{x \in \mathbb{R}^+} \#_w(S \cap xI) > 0$.*

The next result compares qualitative statements about the upper density of a collection of weighted sequences in $\mathbb{R}^+$ with the upper density of the single weighted sequences.

Proposition 4.17. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the following conditions are equivalent.*

(i) *We have $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.*

(ii) *For all $\ell = 1, \ldots, L$, we have $\mathcal{D}^+(S_\ell, w_\ell) < \infty$.*

Proof. First assume that $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$ and fix some $\ell \in \{1, \ldots, L\}$. By definition, there exist $h > 0$ and $N < \infty$ such that $\#_{w_\ell}(S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) < N$ for all $x \in \mathbb{R}^+$. By Proposition 4.15, this implies $\mathcal{D}^+(S_\ell, w_\ell) < \infty$, which yields condition (ii).

Conversely, if $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$, then, by Remark 4.13(b),

$$\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) \leq \sum_{\ell=1}^{L} \mathcal{D}^+(S_\ell, w_\ell) < \infty,$$

hence (i) is satisfied. □

4.5 Affine Density and the Local Integrability Condition

In this section we focus on weighted wavelet systems with finitely many generators of the form

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1)) = \bigcup_{\ell=1}^{L} \{\sqrt{\tfrac{w_\ell(s)}{s}}\psi_\ell(\tfrac{x}{s} - b_\ell k) : s \in S_\ell, k \in \mathbb{Z}\}, \quad (4.4)$$

where $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ are finitely many sequences of arbitrary dilations with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, $b_1, \ldots, b_L > 0$, and $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ are analyzing wavelets. In [77] (compare also Guo and Labate [69] for a correction and some improvements in the situation of classical affine systems in higher dimensions) Hernández, Labate, and Weiss gave a characterization of when a system of the form (4.4) constitutes a Parseval frame for $L^2(\mathbb{R})$, assuming that this system satisfies a certain hypothesis known as the local integrability condition, which is defined as follows.

Definition 4.18. *A system of the form* (4.4) *satisfies the* local integrability condition (LIC)*, if for all* $f \in L^2(\mathbb{R})$ *with* $\hat{f} \in L^\infty(\mathbb{R})$ *and* $\mathrm{supp}\hat{f}$ *being compact in* $\mathbb{R}\backslash\{0\}$*,*

$$I(f) = \sum_{\ell=1}^{L} \frac{1}{b_\ell} \sum_{s\in S_\ell} w_\ell(s) \sum_{m\in\mathbb{Z}} \int_{\mathrm{supp}\hat{f}} |\hat{f}(\xi + \tfrac{m}{sb_\ell})|^2 |\hat{\psi}_\ell(s\xi)|^2 \, d\xi < \infty.$$

We will show that, under some mild regularity assumption on the analyzing wavelets, this rather technical-appearing hypothesis is solely a density condition on the sequence of scale indices. More precisely, in Section 4.5.2 it will be proven that the LIC is equivalent to the condition that the sequences of scale indices possess a finite upper weighted density. This condition is very natural, since every wavelet frame of the form (4.4) must have finite upper weighted density as we saw in Section 4.1. Using this new interpretation of the LIC, in Section 4.5.3 we derive a characterization of wavelet Parseval frames with finitely many generators and with arbitrary dilations provided that the sequences of scale indices possess a finite upper weighted density and the analyzing wavelets belong to an amalgam space which are shown to be very natural hypotheses.

4.5.1 Amalgam Spaces on $\mathbb{R} \setminus \{0\}$

We refer the reader to Section 2.4 for the required notation and a brief overview of the general theory of amalgam spaces. For our purposes, we will need only the following particular amalgam space on the group $\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. The backbone of our definition of the discrete-type norm of this amalgam space is the choice of a collection of subsets of $\mathbb{R}^*$. For each $h > 0$, we let $K_h := (-e^{\frac{h}{2}}, -e^{-\frac{h}{2}}] \cup [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$, and we will use the notation

$K_h(x) = xK_h$ for $x \in \mathbb{R}^*$. It is easily checked that $\{K_1(e^k)\}_{k\in\mathbb{Z}}$ provides us with a tiling of $\mathbb{R}^*$. Using this particular tiling we can define the amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$ on the group $\mathbb{R}^*$ as follows.

Definition 4.19. *A function $f : \mathbb{R} \to \mathbb{C}$ belongs to the* amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$ *if*

$$\|f\|_{W_{\mathbb{R}^*}(L^\infty,L^2)} = \left(\sum_{k\in\mathbb{Z}} \operatorname{ess\,sup}_{x\in K_1(e^k)} |f(x)|^2\right)^{\frac{1}{2}} < \infty.$$

Remark 4.20. We remark that this norm is indeed an equivalent discrete-type norm for the amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$ defined by Definition 2.8. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with $0 \le \phi(x) \le 1$ for all $x \in \mathbb{R}$ satisfying that $\operatorname{supp}(\phi) \subseteq K_2$, $\phi|_{K_1} \equiv 1$, and $\sum_{k\in\mathbb{Z}} \phi(e^k \cdot) \equiv 1$ (this can always be achieved by normalization). Then $\{\phi(e^k \cdot)\}_{k\in\mathbb{Z}}$ forms a BUPU, since $\{e^k\}_{k\in\mathbb{Z}}$ is K_2-dense and relatively separated. Moreover, the norm defined by $\|f\| = \left(\sum_{k\in\mathbb{Z}} \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)\phi(e^k x)|^2\right)^{\frac{1}{2}}$ is equivalent to $\|\cdot\|_{W_{\mathbb{R}^*}(L^\infty,L^2)}$, since $\|\cdot\|^2_{W_{\mathbb{R}^*}(L^\infty,L^2)} \le \|\cdot\|^2 \le 3\,\|\cdot\|^2_{W_{\mathbb{R}^*}(L^\infty,L^2)}$ due to the fact that $K_2(e^k) \subseteq K_1(e^{k-1}) \cup K_1(e^k) \cup K_1(e^{k+1})$ for all $k \in \mathbb{Z}$. Applying Theorem 2.10 now proves the claim.

The following lemma shows that the consideration of analyzing wavelets whose Fourier transform is contained in this amalgam space is by no means restrictive, and is even natural. Specifically, a mild decay condition on $\hat{\psi}$ suffices to ensure that $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

Lemma 4.21. *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Suppose that there exist $a, b, \alpha, \beta > 0$ such that $|\hat{\psi}(\xi)| \le a|\xi|^\alpha$ as $|\xi| \to 0$ and $|\hat{\psi}(\xi)| \le b|\xi|^{-\beta}$ as $|\xi| \to \infty$. Then $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.*

Proof. Let $0 < \omega \le \Omega < \infty$ be such that $|\hat{\psi}(\xi)| \le a|\xi|^\alpha$ for all $|\xi| \le \omega$ and $|\hat{\psi}(\xi)| \le b|\xi|^{-\beta}$ for all $|\xi| \ge \Omega$. Since $\psi \in L^1(\mathbb{R})$, hence $\hat{\psi} \in C(\mathbb{R})$, there exists $M < \infty$ such that $|\hat{\psi}(\xi)| \le M$ for all $|\xi| \in [e^{-1}\omega, e\Omega]$. Then,

$$\begin{aligned}
&\|\hat{\psi}\|^2_{W_{\mathbb{R}^*}(L^\infty,L^2)} \\
&= \sum_{k\in\mathbb{Z}} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2 \\
&\le \sum_{k=-\infty}^{\lfloor \ln\omega - \frac{1}{2}\rfloor} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2 + \sum_{k=\lceil \ln\omega - \frac{1}{2}\rceil}^{\lfloor \ln\Omega + \frac{1}{2}\rfloor} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2 \\
&\quad + \sum_{k=\lceil \ln\Omega + \frac{1}{2}\rceil}^{\infty} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2
\end{aligned}$$

$$\leq a^2 \sum_{k=-\infty}^{\lfloor \ln \omega - \frac{1}{2} \rfloor} (e^{k+\frac{1}{2}})^{2\alpha} + (\ln \Omega - \ln \omega + 2) M^2 + b^2 \sum_{k=\lceil \ln \Omega + \frac{1}{2} \rceil}^{\infty} (e^{k-\frac{1}{2}})^{-2\beta}$$
$$< \infty,$$

which proves the claim. □

Let $\psi \in L^1(\mathbb{R}) \cap L^2_A(\mathbb{R})$. Then it follows that $\hat{\psi}(0) = 0$. If, in addition, the analyzing wavelet ψ possesses a Fourier transform with polynomial decay towards zero and infinity, then $\hat{\psi}$ is contained in $W_{\mathbb{R}^*}(L^\infty, L^2)$.

4.5.2 A Density Version of the Local Integrability Condition

Now we turn to the interpretation of the LIC (see Definition 4.18) in terms of the density of the sequences of scale indices. Our main result gives an equivalent formulation of the LIC in terms of density conditions.

Theorem 4.22. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given, and let $b_1 \ldots, b_L > 0$. Then the following conditions are equivalent.*

(i) *We have $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.*
(ii) *For all $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \ldots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$, the wavelet system $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times b_\ell \mathbb{Z}, (w_\ell, 1))$ satisfies the LIC.*

We will break its proof into several parts to improve clarity. First we derive an easy equivalent formulation of the LIC better suited to our purposes.

Lemma 4.23. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given, and let $b_1 \ldots, b_L > 0$ and $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$. Then the following conditions are equivalent.*

(i) *The system $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times b_\ell \mathbb{Z}, (w_\ell, 1))$ satisfies the LIC.*
(ii) *For all $\ell = 1, \ldots, L$ and $h > 0$,*

$$I_\ell(h) = \frac{1}{b_\ell} \sum_{s \in S_\ell} \frac{w_\ell(s)}{s} \sum_{m \in \mathbb{Z}} \int_{K_h(s) \cap (K_h(s) - \frac{m}{b_\ell})} |\hat{\psi}_\ell(\xi)|^2 \, d\xi < \infty.$$

Proof. (i) ⇒ (ii). Let $I(f)$ be defined as in Definition 4.18. Suppose that (i) holds, i.e., $I(f) < \infty$ for all $f \in L^2(\mathbb{R})$ such that $\hat{f} \in L^\infty(\mathbb{R})$ and $\text{supp}\hat{f}$ is compact in $\mathbb{R} \backslash \{0\}$. Then choosing $f \in L^2(\mathbb{R})$ with $\hat{f} = \chi_{K_h}$ and observing that each of the terms I_ℓ, $\ell = 1, \ldots, L$, is positive, implies (ii).

(ii) ⇒ (i). For $\ell = 1, \ldots, L$ and $h > 0$, we first notice that

$$I_\ell(h) = \frac{1}{b_\ell} \sum_{s \in S_\ell} w_\ell(s) \sum_{m \in \mathbb{Z}} \int_{K_h} \chi_{K_h}(\xi + \tfrac{m}{s b_\ell}) \, |\hat{\psi}_\ell(s\xi)|^2 \, d\xi. \tag{4.5}$$

Now let $f \in L^2(\mathbb{R})$ be such that $\hat{f} \in L^\infty(\mathbb{R})$ and $\text{supp}\hat{f}$ is compact in $\mathbb{R}\backslash\{0\}$. Then there exists $M < \infty$ and a compact set $K \subseteq \mathbb{R}\backslash\{0\}$ such that $|\hat{f}(\xi)| \leq M\chi_K(\xi)$ for almost every $\xi \in \mathbb{R}$. Since $\{K_h\}_{h>0}$ is an exhaustive sequence of compact sets in $\mathbb{R}\backslash\{0\}$, there exists $h > 0$ such that $K \subseteq K_h$. By (ii) and (4.5), this yields

$$\begin{aligned} I(f) &= \sum_{\ell=1}^{L} \frac{1}{b_\ell} \sum_{s\in S_\ell} w_\ell(s) \sum_{m\in\mathbb{Z}} \int_{\text{supp}\hat{f}} |\hat{f}(\xi + \tfrac{m}{sb_\ell})|^2 \, |\hat{\psi}_\ell(s\xi)|^2 \, d\xi \\ &\leq M^2 \sum_{\ell=1}^{L} \frac{1}{b_\ell} \sum_{s\in S_\ell} w_\ell(s) \sum_{m\in\mathbb{Z}} \int_K \chi_K(\xi + \tfrac{m}{sb_\ell}) \, |\hat{\psi}_\ell(s\xi)|^2 \, d\xi \\ &\leq M^2 \sum_{\ell=1}^{L} I_\ell(h) \; < \; \infty. \end{aligned}$$

Thus (i) is satisfied. □

The following lemma establishes a relation between density, the Wiener amalgam space $W_{\mathbb{R}^*}(L^\infty, L^2)$, and a Littlewood–Paley type inequality.

Lemma 4.24. *Let $S \subseteq \mathbb{R}^+$ with associated weight function $w : S \to \mathbb{R}^+$ be given with $\mathcal{D}^+(S, w) < \infty$. Further let $\psi \in L^2(\mathbb{R})$ satisfy $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$. Then there exists $B < \infty$ such that*

$$\sum_{s\in S} w(s)|\hat{\psi}(s\xi)|^2 \leq B \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

Proof. For each $k \in \mathbb{Z}$, set $c_k = \text{ess sup}_{\xi\in K_1(e^k)}|\hat{\psi}(\xi)|^2$. Then we have

$$|\hat{\psi}(\xi)|^2 \leq \sum_{k\in\mathbb{Z}} c_k \chi_{K_1(e^k)}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R} \tag{4.6}$$

and

$$\sum_{k\in\mathbb{Z}} c_k = \|\hat{\psi}\|^2_{W_{\mathbb{R}^*}(L^\infty, L^2)}. \tag{4.7}$$

Since $S \subseteq \mathbb{R}^+$, equation (4.6) yields

$$\begin{aligned} \text{ess sup}_{\xi\in\mathbb{R}^*} \sum_{s\in S} w(s)|\hat{\psi}(s\xi)|^2 &\leq \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S} w(s) \sum_{k\in\mathbb{Z}} c_k \, \chi_{K_1(e^k)}(s\xi) \\ &\leq \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S} w(s)\chi_{K_1(\xi^{-1}e^k)}(s) \\ &= \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^*} \sum_{s\in S} w(s)\chi_{K_1(\xi)}(s) \\ &= \sum_{k\in\mathbb{Z}} c_k \sup_{\xi\in\mathbb{R}^+} \#_w(S \cap \xi[e^{-\frac{1}{2}}, e^{\frac{1}{2}})). \end{aligned}$$

Since $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ and $\mathcal{D}^+(S, w) < \infty$, the last quantity is a finite constant by equation (4.7) and Proposition 4.15. □

In the following lemma, by using a sequence in $\mathbb{R}^+$, we explicitly construct functions whose Fourier transforms are contained in $W_{\mathbb{R}^*}(L^\infty, L^2)$.

Lemma 4.25. *Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be such that the sets $y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})$, $n \in \mathbb{N}$, are mutually disjoint.*

(i) *Suppose that $y_n \to 0$ as $n \to \infty$. Then the function $\psi \in L^2(\mathbb{R})$ defined by*

$$\hat{\psi} = \sum_{n\in\mathbb{N}} \frac{1}{n} \chi_{y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})}$$

satisfies $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

(ii) *Suppose that $y_n \to \infty$ as $n \to \infty$. Then the function $\psi \in L^2(\mathbb{R})$ defined by*

$$\hat{\psi} = \sum_{n\in\mathbb{N}} \frac{1}{n\sqrt{y_n}} \chi_{y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})}$$

satisfies $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

Proof. (i) Suppose that $y_n \to 0$ as $n \to \infty$. It is easy to check that $\hat{\psi} \in L^2(\mathbb{R})$, hence $\psi \in L^2(\mathbb{R})$. We next observe that for each $k \in \mathbb{Z}$ and $x \in \mathbb{R}^+$, we have $e^k[e^{-\frac{1}{2}}, e^{\frac{1}{2}}) \cap x[e^{-\frac{1}{2}}, e^{\frac{1}{2}}) \neq \emptyset$ if and only if $\ln x - 1 \le k \le \ln x + 1$. Therefore we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sup_{\xi\in K_1(e^k)} |\hat{\psi}(\xi)|^2 &\le \sum_{n\in\mathbb{N}} \frac{1}{n^2} \sum_{k\in\mathbb{Z}} \sup_{\xi\in e^k[e^{-\frac{1}{2}}, e^{\frac{1}{2}})} \chi_{y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})}(\xi) \\
&= \sum_{n\in\mathbb{N}} \frac{1}{n^2} \#\{k \in \mathbb{Z} : e^k[e^{-\frac{1}{2}}, e^{\frac{1}{2}}) \cap y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}}) \neq \emptyset\} \\
&\le 3 \sum_{n\in\mathbb{N}} \frac{1}{n^2} \;<\; \infty,
\end{aligned}$$

hence $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$.

(ii) Now suppose that $y_n \to \infty$ as $n \to \infty$. The slightly different definition of ψ ensures that also in this case $\hat{\psi} \in L^2(\mathbb{R})$, hence $\psi \in L^2(\mathbb{R})$. Then $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ can be proven in a similar way as in part (i). □

Now we are prepared to prove Theorem 4.22.

Proof (of Theorem 4.22). (i) ⇒ (ii). We suppose that $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$, which by Proposition 4.17 implies that $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$. For arbitrary functions $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \ldots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$, we have to show that $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1))$ satisfies the LIC. As observed in Lemma 4.23, it suffices to prove that

$$I_\ell(h) = \frac{1}{b_\ell} \sum_{s\in S_\ell} \frac{w_\ell(s)}{s} \sum_{m\in\mathbb{Z}} \int_{K_h(s)\cap(K_h(s)-\frac{m}{b_\ell})} |\hat{\psi}_\ell(\xi)|^2 \, d\xi < \infty, \tag{4.8}$$

for all $\ell = 1, \ldots, L$ and $h > 0$. For this, fix $h > 0$, $\psi \in L^2(\mathbb{R})$ with $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$, and consider some $\ell \in \{1, \ldots, L\}$. For the sake of brevity, we set $I(h) = I_\ell(h)$, $S = S_\ell$, $w = w_\ell$, and $b = b_\ell$. We decompose $I(h)$ by

$$I(h) = I_1(h) + I_2(h),$$

where

$$I_1(h) = \frac{1}{b} \sum_{s \in S} \frac{w(s)}{s} \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi$$

and

$$I_2(h) = \frac{1}{b} \sum_{s \in S} \frac{w(s)}{s} \sum_{m \in \mathbb{Z} \backslash \{0\}} \int_{K_h(s) \cap (K_h(s) - \frac{m}{b})} |\hat{\psi}(\xi)|^2 \, d\xi.$$

First, we study $I_1(h)$. By Lemma 4.24, there exists some $B < \infty$ such that

$$\sum_{s \in S} w(s) |\hat{\psi}(s\xi)|^2 \leq B \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} I_1(h) &= \frac{1}{b} \sum_{s \in S} \frac{w(s)}{s} \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi \\ &= \frac{1}{b} \int_{K_h} \sum_{s \in S} w(s) |\hat{\psi}(s\xi)|^2 \, d\xi \leq \frac{1}{b} B |K_h| < \infty. \end{aligned} \tag{4.9}$$

Secondly, we show that $I_2(h)$ is finite. Let $s \in S$ be fixed. We observe that if $s[e^{-\frac{h}{2}}, e^{\frac{h}{2}}) \cap (s[e^{-\frac{h}{2}}, e^{\frac{h}{2}}) - \frac{m}{b}) \neq \emptyset$, then

$$se^{-\frac{h}{2}} \leq se^{\frac{h}{2}} - \tfrac{m}{b} \quad \text{and} \quad se^{\frac{h}{2}} \geq se^{-\frac{h}{2}} - \tfrac{m}{b}.$$

This is equivalent to

$$-sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \leq m \leq sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}).$$

By the choice of $K_h(s)$, for each $s \in S$ there exist at most $3(2sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) + 1)$ integers m such that $K_h(s) \cap (K_h(s) - \frac{m}{b}) \neq \emptyset$. Recall that in this case we only consider $m \in \mathbb{Z} \backslash \{0\}$. Therefore there exists $\varepsilon > 0$ such that $K_h(s) \cap (K_h(s) - \frac{m}{b}) = \emptyset$ for all $s \in S$ with $s < \varepsilon$ and $m \in \mathbb{Z} \backslash \{0\}$. This shows that we only need to consider those $s \in S$ with $s \geq \varepsilon$. Then there exists a $C < \infty$ such that

$$3(2sb(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) + 1) \leq Cs(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \text{ for all } s \in S, \ s \geq \varepsilon.$$

Using these observations, we obtain

$$\begin{aligned} I_2(h) &\leq \frac{1}{b} \sum_{s \in S} \frac{w(s)}{s} Cs(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi \\ &= \frac{C}{b} (e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \sum_{s \in S} w(s) \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi. \end{aligned} \tag{4.10}$$

It follows easily from $\mathcal{D}^+(S,w) < \infty$ that there exists an $N < \infty$ such that

$$\#_w\{s \in S : x \in K_h(s)\} \le N \quad \text{for all } x \in \mathbb{R}.$$

Continuing equation (4.10), we obtain

$$I_2(h) \le \frac{C}{b}(e^{\frac{h}{2}} - e^{-\frac{h}{2}}) \sum_{s\in S} w(s) \int_{K_h(s)} |\hat{\psi}(\xi)|^2 \, d\xi \le \frac{C}{b}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})\, N \,\|\hat{\psi}\|_2^2 < \infty. \tag{4.11}$$

Combining the estimates (4.9) and (4.11) yields (4.8). Thus (ii) holds.

(ii) ⇒ (i). Suppose that (ii) holds. Towards a contradiction assume that we have $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) = \infty$. By Proposition 4.17, there exists $\ell_0 \in \{1,\dots,L\}$ such that $\mathcal{D}^+(S_{\ell_0}, w_{\ell_0}) = \infty$. Thus, by Lemma 4.23, to obtain a contradiction it suffices to show that there exists $\psi \in L^2(\mathbb{R})$ with $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ such that for some $h > 0$,

$$I_{\ell_0}(h) = \frac{1}{b_{\ell_0}} \sum_{s\in S_{\ell_0}} \frac{w_{\ell_0}(s)}{s} \sum_{m\in\mathbb{Z}} \int_{K_h(s)\cap(K_h(s) - \frac{m}{b_{\ell_0}})} |\hat{\psi}(\xi)|^2 \, d\xi = \infty. \tag{4.12}$$

To simplify notation we set $I(h) = I_{\ell_0}(h)$, $S = S_{\ell_0}$, $w = w_{\ell_0}$, and $b = b_{\ell_0}$. Proposition 4.15 implies the existence of sequences $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ and $\{S_n\}_{n\in\mathbb{N}}$ with $S_n \subseteq S$ satisfying that $\#_w(S_n) \ge n$ and $S_n \subseteq y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})$.

If there exists $y \in \mathbb{R}^+$ and $h > 0$ with $\#_w(S \cap y[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) = \infty$, then choosing $\psi \in L^2(\mathbb{R})$ by $\hat{\psi} = \chi_{y[e^{-\frac{h}{2}}, e^{\frac{h}{2}})K_h} \in W_{\mathbb{R}^*}(L^\infty, L^2)$ yields

$$\begin{aligned} I(h) &= \frac{1}{b} \sum_{s\in S} \frac{w(s)}{s} \sum_{m\in\mathbb{Z}} \int_{K_h(s)\cap(K_h(s)-\frac{m}{b})} |\hat{\psi}(\xi)|^2 \, d\xi \\ &\ge \frac{1}{b} \sum_{s\in S\cap y[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} \frac{w(s)}{s} \int_{K_h(s)} \chi_{y[e^{-\frac{h}{2}}, e^{\frac{h}{2}})K_h}(\xi)\, d\xi \\ &= \frac{1}{b} \sum_{s\in S\cap y[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} \frac{w(s)}{s} s\, |K_h| = \infty. \end{aligned}$$

This settles (4.12) for the chosen h.

Otherwise we remark that, by restricting $\{y_n\}_{n\in\mathbb{N}}$ to a subsequence if necessary, we have either $y_n \to 0$ or $y_n \to \infty$ as $n \to \infty$. Moreover, without loss of generality we may assume that the sets $y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})$, $n \in \mathbb{N}$, are mutually disjoint by choosing again an appropriate subsequence if necessary.

First assume that $y_n \to 0$ as $n \to \infty$. Then we define the function $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi} = \sum_{n\in\mathbb{N}} \frac{1}{n} \chi_{y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})}.$$

Lemma 4.25(i) implies that $\hat{\psi} \in W_{\mathbb{R}^*}(L^\infty, L^2)$. Choosing $h = 2$, we obtain

$$I(2) = \frac{1}{b}\sum_{s\in S}\frac{w(s)}{s}\sum_{m\in\mathbb{Z}}\int_{K_2(s)\cap(K_2(s)-\frac{m}{b})}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\chi_{y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})}(\xi)\,d\xi$$
$$\geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{w(s)}{s}\sum_{m\in\mathbb{Z}}|s[e^{-1},e)\cap(s[e^{-1},e)-\tfrac{m}{b})$$
$$\cap y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})|. \tag{4.13}$$

Since $S_n \subseteq y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})$, it follows that for each $s\in S_n$,

$$y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}}) = [y_n e^{-\frac{1}{2}}, y_n e^{\frac{1}{2}}) \subseteq [se^{-\frac{1}{2}}e^{-\frac{1}{2}}, se^{\frac{1}{2}}e^{\frac{1}{2}}) = s[e^{-1},e). \tag{4.14}$$

This implies that $\frac{y_n}{s}[e^{-\frac{1}{2}},e^{\frac{1}{2}}) \subseteq [e^{-1},e)$, and hence an easy computation shows that $\frac{y_n}{s}\in[e^{-\frac{1}{2}},e^{\frac{1}{2}})$. Thus

$$|[e^{-1},e)\cap \tfrac{y_n}{s}[e^{-\frac{1}{2}},e^{\frac{1}{2}})| \geq 1-e^{-1} \quad \text{for all } n\in\mathbb{N}, s\in S_n. \tag{4.15}$$

Therefore, employing (4.14) and (4.15), we can continue the computation in (4.13) to obtain

$$I(2) \geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{w(s)}{s}\sum_{m\in\mathbb{Z}}|s[e^{-1},e)\cap(s[e^{-1},e)-\tfrac{m}{b})\cap y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})|$$
$$\geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}\frac{w(s)}{s}|s[e^{-1},e)\cap y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})|$$
$$= \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{s\in S_n}w(s)|[e^{-1},e)\cap \tfrac{y_n}{s}[e^{-\frac{1}{2}},e^{\frac{1}{2}})|$$
$$\geq \frac{1-e^{-1}}{b}\sum_{n\in\mathbb{N}}\frac{1}{n} = \infty.$$

This settles (4.12) for $h=2$.

Secondly, assume that $y_n\to\infty$ as $n\to\infty$. In this case we define $\psi\in L^2(\mathbb{R})$ by

$$\hat{\psi} = \sum_{n\in\mathbb{N}}\frac{1}{n\sqrt{y_n}}\chi_{y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})}.$$

Lemma 4.25(ii) implies that $\hat{\psi}\in W_{\mathbb{R}^*}(L^\infty,L^2)$. We further observe that

$$s[e^{-1},e)\subseteq s[e^{-2},e^2)-\tfrac{m}{b} \tag{4.16}$$

if and only if $-sb(e^{-1}-e^{-2}) \leq m \leq sb(e^2-e)$. Thus there exist at least $sb(e^2-e+e^{-1}-e^{-2}) = sbC'$ values of m for which (4.16) is true. Choosing $h=4$ and using (4.14) yields

$$I(4) \geq \frac{1}{b}\sum_{n\in\mathbb{N}}\frac{1}{n^2 y_n}\sum_{s\in S_n}\frac{w(s)}{s}\sum_{m\in\mathbb{Z}}|s[e^{-2},e^2)\cap(s[e^{-2},e^2)-\tfrac{m}{b})\cap y_n[e^{-\frac{1}{2}},e^{\frac{1}{2}})|$$

$$\geq \frac{1}{b} \sum_{n\in\mathbb{N}} \frac{1}{n^2 y_n} \sum_{s\in S_n} \frac{w(s)}{s} sbC' |y_n[e^{-\frac{1}{2}}, e^{\frac{1}{2}})|$$

$$= C'(e^{\frac{1}{2}} - e^{-\frac{1}{2}}) \sum_{n\in\mathbb{N}} \frac{1}{n^2} \sum_{s\in S_n} w(s)$$

$$\geq C'(e^{\frac{1}{2}} - e^{-\frac{1}{2}}) \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty.$$

This settles (4.12) for $h = 4$.

Hence (ii) is not satisfied, a contradiction. □

4.5.3 A Characterization of Wavelet Parseval Frames

The equivalent formulation of the LIC in terms of density conditions yields the following characterization result for weighted wavelet Parseval frames with finitely many generators and with arbitrary sequences of scale indices.

Theorem 4.26. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Suppose that $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$. Then for all $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ with $\hat{\psi}_1, \ldots, \hat{\psi}_L \in W_{\mathbb{R}^*}(L^\infty, L^2)$ and $b_1, \ldots, b_L > 0$, the following conditions are equivalent.*

(i) *$\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1))$ is a Parseval frame for $L^2(\mathbb{R})$.*

(ii) *For each $\alpha \in \bigcup_{\ell=1}^L \bigcup_{s\in S_\ell} \frac{1}{b_\ell s}\mathbb{Z}$, where $\mathcal{P}_\alpha = \{(\ell, s) \in \{1, \ldots, L\} \times S_\ell : b_\ell s\alpha \in \mathbb{Z}\}$, we have*

$$\sum_{(\ell,s)\in\mathcal{P}_\alpha} \frac{w_\ell(s)}{b_\ell} \overline{\hat{\psi}_\ell(s\xi)} \hat{\psi}_\ell(s(\xi + \alpha)) = \delta_{\alpha,0} \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

Proof. The claim follows immediately from Theorem 4.22 and [77, Thm. 2.1]. □

At last, we show that the hypothesis of finite upper weighted density is not at all restrictive.

Proposition 4.27. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given, and let $b_1, \ldots, b_L > 0$ and $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$. Then Theorem 4.26(i) implies $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$, and if, in addition, $\psi_1, \ldots, \psi_L \in L^1(\mathbb{R})$, then also Theorem 4.26(ii) implies $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.*

Proof. First suppose Theorem 4.26(i) holds, i.e., $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1))$ is a Parseval frame for $L^2(\mathbb{R})$. Then, in particular, for each $\ell = 1, \ldots, L$, the wavelet system $\mathcal{W}(\psi_\ell, S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1))$ is a Bessel sequence, i.e., it possesses an upper frame bound. Now Theorem 4.1 implies that $\mathcal{D}^+(S_\ell \times b_\ell\mathbb{Z}, (w_\ell, 1)) <$

∞ for all $\ell = 1, \ldots, L$. A simple computation shows that this implies $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$. The application of Proposition 4.17 then proves the first claim.

Secondly, suppose that $\psi_1, \ldots, \psi_L \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and Theorem 4.26(ii) holds. Notice that $\mathcal{P}_0 = \{(\ell, s) \in \{1, \ldots, L\} \times S_\ell\}$. Hence in the special case $\alpha = 0$, we obtain

$$\sum_{\ell=1}^{L} \frac{1}{b_\ell} \sum_{s \in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}. \tag{4.17}$$

Towards a contradiction assume that there exists $\ell_0 \in \{1, \ldots, L\}$ with $\mathcal{D}^+(S_{\ell_0}, w_{\ell_0}) = \infty$. Since $\hat{\psi}_{\ell_0}$ is continuous, there exists some interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$ such that $|\hat{\psi}_{\ell_0}(\xi)|^2 \geq \delta > 0$ for all $\xi \in I$. Applying Proposition 4.15, for each $n \in \mathbb{N}$, there exists some $y_n \in \mathbb{R}$ with $\#_{w_{\ell_0}}(S_{\ell_0} \cap y_n I) \geq n$. Hence, for all $n \in \mathbb{N}$,

$$\sum_{s \in S_{\ell_0}} w_{\ell_0}(s) |\hat{\psi}_{\ell_0}(s y_n^{-1})|^2 \geq \sum_{s \in S_{\ell_0} \cap y_n I} w_{\ell_0}(s) |\hat{\psi}_{\ell_0}(s y_n^{-1})|^2 \geq \delta\, n,$$

a contradiction to (4.17). Thus $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ for all $\ell = 1, \ldots, L$. Proposition 4.17 then settles the claim. □

5

Quantitative Density Conditions

In this chapter we study weighted irregular wavelet frames with finitely many generators. We derive a fundamental relationship between the affine weighted density, the frame bounds, and the admissibility constants for the analyzing wavelets.

Then several applications of this result are discussed. In particular, we derive that the affine density of a tight wavelet frame necessarily has to be uniform. Further, our results reveal one reason why there does not exist a Nyquist phenomenon for wavelet systems. We also study the extent to which affine density conditions can serve as sufficient conditions for the existence of wavelet frames thereby presenting a situation, where this indeed can be achieved. Finally, we use the fundamental relationship to prove that certain weighted co-affine systems can never form a frame.

The main results in this chapter, except the application to co-affine systems, generalize the results obtained in Kutyniok [95] to multiple generators. This chapter also contains a generalization of one result in Heil and Kutyniok [74].

5.1 Outline and Comparison with Previous Work

Provided that an analyzing wavelet $\psi \in L^2(\mathbb{R})$ gives rise to a classical affine frame $\{a^{-j/2}\psi(a^{-j}x - bk)\}_{j,k\in\mathbb{Z}}$ with parameters $a > 1$, $b > 0$ and with frame bounds A, B, a result by Chui and Shi [33] and by Daubechies [41] establishes the following intriguing relationship between the parameters, the frame bounds, and the admissibility constant C_ψ:

$$A \le \frac{1}{2b\ln a}\, C_\psi \le B. \tag{5.1}$$

In particular, this leads to the exact value of the frame bound for tight classical affine frames in terms of the parameters and the admissibility constant.

In this chapter our driving motivation is to derive necessary and sufficient density conditions not merely on the sequence of time-scale indices for irregular wavelet frames, but also involving the explicit values of the frame bounds, in this sense *quantitative* density conditions. For our study we will focus on weighted irregular wavelet systems with respect to the weighted sequences of scale-time indices $(S_\ell \times T_\ell, (w_\ell, v_\ell))$, where $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ are arbitrary sequences of scale indices equipped with weights $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ and $T_1, \ldots, T_L \subseteq \mathbb{R}$ are arbitrary sequences of time indices equipped with weights $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, and finitely many wavelets $\psi_1, \ldots, \psi_L \in L^2_A(\mathbb{R})$. That means we consider wavelet systems of the form $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$. In Section 5.3 we derive a very general relation between the weighted affine density, the frame bounds, and the admissibility constants for the analyzing wavelets (Theorem 5.6). This result can indeed be shown to contain (5.1) as a special case.

The proof of Theorem 5.6 requires several preliminary results, each of which is very interesting in its own right. We first show that wavelet frames of the form $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ impose a Littlewood–Paley type relation on the sequences of scale indices and on the analyzing wavelets (Proposition 5.3), thereby generalizing a result by Yang and Zhou [127] to weighted wavelet systems with finitely many generators. These Littlewood–Paley type inequalities can be related to density conditions on the associated sequence and a constant depending on the involved function comparable with the admissibility constant (Proposition 5.4). In Theorem 5.5 we further derive a new relationship between the density, the frame bounds, and the norms of the generators of a frame of weighted exponentials, which contains a result from Heil and Kutyniok [74] as a special case. These results then serve as the main ingredients in the proof of Theorem 5.6. This main result implies that the affine density of a tight wavelet frame necessarily has to be uniform. Several other intriguing applications will then be shown in Sections 5.4–5.7. In the remainder of this section we will discuss those briefly.

Interestingly, our main result has direct impact on the well-known question initially stated by Daubechies [41, Sec. 4.1], namely, why wavelet systems do not satisfy a Nyquist phenomenon analogous to Gabor systems. Our result now reveals one reason why there does not exist a critical density for orthonormal wavelet bases. In brief, the sequence of time-scale indices of an orthonormal wavelet basis has indeed uniform affine density. However, Theorem 5.6 implies that for these systems

$$\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda) = 2\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\xi}{|\xi|})},$$

which can be shown to attain each positive value. A detailed analysis is presented in Section 5.4.

Conceptually, density conditions seem to be capable only of delivering necessary conditions for the existence of wavelet frames, since they are independent of the analyzing wavelet itself and do not capture local features of the sequence of time-scale indices. Hence they appear almost too weak to serve as a sufficient condition. And in fact, to date, the notion of density was only employed to derive necessary conditions. However, in Section 5.5 we show that under some mild decay condition on the analyzing wavelets, the existence of a sequence of time indices T such that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T, (w_\ell, 1))$ forms a frame is in fact equivalent to the weighted sequence $\bigcup_{\ell=1}^{L} S_\ell$ having positive lower and finite upper weighted density (Theorem 5.11). Section 5.6 is then devoted to the study whether such weight functions always exist.

In the last section, we consider co-affine systems with arbitrary sequences of time-scale indices with mild conditions on the sequences of time indices and the analyzing wavelets, and show by employing previously derived results that such systems can never form a frame (Theorem 5.18). This is indeed a generalization to arbitrary sequences of scale indices of a result derived by Gressman, Labate, Weiss, and Wilson [61]. Thus, in particular, interchanging the dilation and translation operator for systems of the form $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, 1))$ results in the complete loss of the frame properties, thereby indicating how sensitive discrete wavelet frames behave with respect to the ordering of the operators in contrast to continuous wavelet frames.

5.2 Density of Product Sequences

Since in the sequel we will study sequences in $\mathbb{A}$ of the form $\Lambda = S \times T$, where $S \subseteq \mathbb{R}^+$ and $T \subseteq \mathbb{R}$, the definition of density for sequences in $\mathbb{R}^+$ and $\mathbb{R}$ will become important. Notice that we use the same notion for all three densities. The type of density is then always completely determined by the sequence to which it is applied.

The notion of density for sequences in $\mathbb{R}^+$ has already been introduced in Section 4.4. The notion of Beurling density for weighted sequences in $\mathbb{R}^d$ will be introduced in Chapter 7. Since in this chapter we are only concerned with sequences in $\mathbb{R}$, for the convenience of the reader we will state the new definition of weighted Beurling density for this special case.

For $T_1, \ldots, T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, the *upper weighted Beurling density* of $\{(T_\ell, v_\ell)\}_{\ell=1}^{L}$ is defined by

$$\mathcal{D}^+(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}} \frac{\sum_{\ell=1}^{L} \#_{v_\ell}(T_\ell \cap x + [-\frac{h}{2}, \frac{h}{2}))}{h},$$

and the *lower weighted Beurling density* of $\{(T_\ell, v_\ell)\}_{\ell=1}^{L}$ is

$$\mathcal{D}^-(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}} \frac{\sum_{\ell=1}^{L} \#_{v_\ell}(T_\ell \cap x + [-\frac{h}{2}, \frac{h}{2}))}{h}.$$

If we have $\mathcal{D}^-(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) = \mathcal{D}^+(\{(T_\ell, v_\ell)\}_{\ell=1}^{L})$, then $\{(T_\ell, v_\ell)\}_{\ell=1}^{L}$ is said to possess the *uniform weighted Beurling density* $\mathcal{D}(\{(T_\ell, v_\ell)\}_{\ell=1}^{L})$.

For $S \subseteq \mathbb{R}^+$ with $w : S \to \mathbb{R}^+$ and $T \subseteq \mathbb{R}$ with $v : T \to \mathbb{R}^+$, we will define the weight function $(w, v) : S \times T \to \mathbb{R}^+$ by $(w, v)(s, t) = w(s)v(t)$. Under some mild density conditions on $T_1, \dots, T_L \subseteq \mathbb{R}$ equipped with weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \dots, L$, the affine density of $\{(S_\ell \times T_\ell, (w_\ell, v_\ell))\}_{\ell=1}^L$ can be computed in the following way.

Lemma 5.1. *Let $S_1, \dots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \dots, L$, and $T_1, \dots, T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \dots, L$ be given. If for each $\ell = 1, \dots, L$ the pair (T_ℓ, v_ℓ) possesses a uniform weighted Beurling density $\mathcal{D}(T_\ell, v_\ell)$, then*

$$\mathcal{D}^-(\{(S_\ell \times T_\ell, (w_\ell, v_\ell))\}_{\ell=1}^L) = \mathcal{D}^-(\{(S_\ell, \mathcal{D}(T_\ell, v_\ell) \cdot w_\ell)\}_{\ell=1}^L)$$

and

$$\mathcal{D}^+(\{(S_\ell \times T_\ell, (w_\ell, v_\ell))\}_{\ell=1}^L) = \mathcal{D}^+(\{(S_\ell, \mathcal{D}(T_\ell, v_\ell) \cdot w_\ell)\}_{\ell=1}^L).$$

Proof. Fix $\varepsilon > 0$. Since (T_ℓ, v_ℓ) possesses a uniform weighted Beurling density, there exists $h_0 > 0$ with

$$\left| \frac{\#_{v_\ell}(T_\ell \cap x + [-\frac{h}{2}, \frac{h}{2}))}{h} - \mathcal{D}(T_\ell, v_\ell) \right| < \varepsilon \quad \text{for all } x \in \mathbb{R},\ h \geq h_0, \ell = 1, \dots, L. \tag{5.2}$$

Set $\Lambda_\ell = S_\ell \times T_\ell$. For each $(x, y) \in \mathbb{A}$,

$$\begin{aligned}
&\sum_{\ell=1}^L \#_{(w_\ell, v_\ell)}(\Lambda_\ell \cap Q_h(x, y)) \\
&= \sum_{\ell=1}^L \#_{(w_\ell, v_\ell)}(\Lambda_\ell \cap \{(xa, \tfrac{y}{a} + b) : a \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}}),\, b \in [-\tfrac{h}{2}, \tfrac{h}{2})\}) \\
&= \sum_{\ell=1}^L \sum_{s \in S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} w_\ell(s) \cdot \#_{v_\ell}(T_\ell \cap \tfrac{xy}{s} + [-\tfrac{h}{2}, \tfrac{h}{2})).
\end{aligned}$$

Dividing by h^2, taking the infimum over all $(x, y) \in \mathbb{A}$, and employing (5.2), yields

$$\begin{aligned}
&\inf_{x \in \mathbb{R}^+} \frac{\sum_{\ell=1}^L \#_{(\mathcal{D}(T_\ell, v_\ell) - \varepsilon) \cdot w_\ell}(S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))}{h} \\
&\leq \inf_{(x,y) \in \mathbb{A}} \frac{\sum_{\ell=1}^L \#_{(w_\ell, v_\ell)}(\Lambda_\ell \cap Q_h(x, y))}{h^2} \\
&\leq \inf_{x \in \mathbb{R}^+} \frac{\sum_{\ell=1}^L \#_{(\mathcal{D}(T_\ell, v_\ell) + \varepsilon) \cdot w_\ell}(S_\ell \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))}{h}
\end{aligned}$$

for all $h \geq h_0$. Applying the liminf as $h \to \infty$ and noting that we can choose ε arbitrarily small, proves $\mathcal{D}^-(\{(S_\ell \times T_\ell, (w_\ell, v_\ell))\}_{\ell=1}^L) = \mathcal{D}^-(\{(S_\ell, \mathcal{D}(T_\ell, v_\ell) \cdot w_\ell)\}_{\ell=1}^L)$.

The second claim can be treated similarly. □

5.3 A Fundamental Relationship

For the proof of the fundamental relationship between the affine density of the sequences of time and scale indices, the frame bounds, and the admissibility constants of the analyzing wavelets, we need to consider the system of weighted exponentials associated with some weighted sequence of time indices. Given $r > 0$, $T \subseteq \mathbb{R}$, and $v : T \to \mathbb{R}^+$, we denote the corresponding system of weighted exponentials by

$$\mathcal{E}(T, v, r) = \{x \mapsto v(t)^{\frac{1}{2}} e^{2\pi i t x} : t \in T, x \in [-r, r]\}.$$

Then $\mathcal{E}(T, v, r)$ is a frame for $L^2[-r, r]$ with frame bounds A and B, if for all $f \in L^2[-r, r]$,

$$A \int_{-r}^{r} |f(x)|^2 \, dx \le \sum_{t \in T} v(t) \left| \int_{-r}^{r} f(x) e^{-2\pi i t x} \, dx \right|^2 \le B \int_{-r}^{r} |f(x)|^2 \, dx.$$

We first show that for the wavelet systems under consideration, a finite upper weighted density of the weighted sequences of scale indices is a necessary condition for such a system to form a Bessel sequence. This result generalizes the result by Sun and Zhou [121, Thm 2.1(1)] to weighted wavelet systems. We remark that this proposition is not a direct corollary from Theorem 4.1.

Proposition 5.2. *Let $\psi_1, \ldots, \psi_L \subseteq L^2(\mathbb{R}) \backslash \{0\}$, $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, and $T_1, \ldots, T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given.*

(i) *If $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ is a Bessel sequence for $L^2(\mathbb{R})$, then we have $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^{L}) < \infty$.*

(ii) *If $\bigcup_{\ell=1}^{L} \mathcal{E}(T_\ell, v_\ell, r)$, where $r > 0$, is a Bessel sequence for $L^2[-r, r]$, then we have $\mathcal{D}^+(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) < \infty$.*

Proof. (i) Let B denote the Bessel bound for $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$. Fix $\ell \in \{1, \ldots, L\}$, $s_0 \in S_\ell$, and $t_0 \in T_\ell$. Let $f = \sigma(s_0, t_0)\psi_\ell$. Now since $W_{\psi_\ell} f$ is continuous and $W_{\psi_\ell} f(s_0, t_0) \neq 0$, there must exist some $h > 0$ such that

$$\inf_{x \in [e^{-\frac{h}{2}}, e^{\frac{h}{2}})} |W_{\psi_\ell} f(x s_0, t_0)| = \delta > 0.$$

Then for all $y \in \mathbb{R}^+$,

$$\begin{aligned}
B \|f\|_2^2 &= B \left\| \sigma(s_0^{-1} y, 0) f \right\|_2^2 \\
&\ge \sum_{s \in S_\ell} w_\ell(s) \sum_{t \in T_\ell} v_\ell(t) \left| \left\langle \sigma(s_0^{-1} y, 0) f, \sigma(s, t)\psi_\ell \right\rangle \right|^2 \\
&\ge v_\ell(t_0) \sum_{s \in S_\ell} w_\ell(s) \left| \left\langle \sigma(s_0^{-1} y, 0) f, \sigma(s, t_0)\psi_\ell \right\rangle \right|^2
\end{aligned}$$

$$\geq v_\ell(t_0) \sum_{s\in S_\ell\cap y[e^{-\frac{h}{2}},e^{\frac{h}{2}})} w_\ell(s)\left|\langle f,\sigma(sy^{-1}s_0,t_0)\psi_\ell\rangle\right|^2$$
$$\geq v_\ell(t_0)\delta^2\#_{w_\ell}(S_\ell\cap y[e^{-\frac{h}{2}},e^{\frac{h}{2}})).$$

By Propositions 4.15 and 4.17, the claim follows.

(ii) The proof is similar to the proof of part (i). □

Next we show that wavelet frames of the form considered in this chapter impose a Littlewood–Paley type relation on the sequences of scale indices and on the analyzing wavelets. This result extends the result by Yang and Zhou [127, Thm. 1] to weighted wavelet systems with finitely many generators.

Proposition 5.3. *Let $S_1,\dots,S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1,\dots,L$, and $T_1,\dots,T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1,\dots,L$ be given. Further, let $\psi_1,\dots,\psi_L \subseteq L^2_A(\mathbb{R})\backslash\{0\}$. If $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell\times T_\ell,(w_\ell,v_\ell))$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B, and if for all $\ell = 1,\dots,L$ the system $\mathcal{E}(T_\ell,v_\ell,r)$ is a frame for $L^2[-r,r]$ with frame bounds C_ℓ and D_ℓ for some $r > 0$, then, setting $C_{\min} = \min_{\ell=1,\dots,L} C_\ell$ and $D_{\max} = \max_{\ell=1,\dots,L} D_\ell$, we have*

$$\frac{A}{D_{\max}} \leq \sum_{\ell=1}^L\sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2 \leq \frac{B}{C_{\min}} \quad \text{for a.e. } \xi\in\mathbb{R}. \tag{5.3}$$

Proof. Let $f \in L^2(\mathbb{R})$, $\xi_0\in\mathbb{R}$, and $M > 0$. By Lemma 2.2, we have

$$\begin{aligned}
&\sum_{\ell=1}^L\sum_{s\in S_\ell}\sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\left|\langle f,\sigma(s,t)\psi_\ell\rangle\right|^2\\
&= \sum_{\ell=1}^L\sum_{s\in S_\ell}\sum_{t\in T_\ell} w_\ell(s)v_\ell(t)s\left|\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}_\ell(s\xi)}e^{2\pi ist\xi}\,d\xi\right|^2\\
&= \sum_{\ell=1}^L\sum_{s\in S_\ell}\sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\frac{1}{s}\left|\int_{-\infty}^{\infty}\hat{f}(\tfrac{\xi}{s})\overline{\hat{\psi}_\ell(\xi)}e^{2\pi it\xi}\,d\xi\right|^2\\
&= \sum_{\ell=1}^L\sum_{s\in S_\ell}\sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\frac{1}{s}\left|\int_{s\xi_0-r}^{s\xi_0+r}\sum_{k\in\mathbb{Z}}\hat{f}(\tfrac{\xi+2kr}{s})\overline{\hat{\psi}_\ell(\xi+2kr)}e^{2\pi it(\xi+2kr)}\,d\xi\right|^2\\
&= J_1(f,M,\xi_0)+J_2(f,M,\xi_0),
\end{aligned}$$

where

$$J_1(f,M,\xi_0) = \sum_{\ell=1}^{L} \sum_{s\in S_\ell, s\le M} \sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\frac{1}{s}$$
$$\cdot \left| \int_{s\xi_0-r}^{s\xi_0+r} \sum_{k\in\mathbb{Z}} \hat{f}(\tfrac{\xi+2kr}{s})\overline{\hat{\psi}_\ell(\xi+2kr)}e^{2\pi it(\xi+2kr)}\,d\xi \right|^2 \quad \text{and}$$
$$J_2(f,M,\xi_0) = \sum_{\ell=1}^{L} \sum_{s\in S_\ell, s> M} \sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\frac{1}{s}$$
$$\cdot \left| \int_{s\xi_0-r}^{s\xi_0+r} \sum_{k\in\mathbb{Z}} \hat{f}(\tfrac{\xi+2kr}{s})\overline{\hat{\psi}_\ell(\xi+2kr)}e^{2\pi it(\xi+2kr)}\,d\xi \right|^2 .$$

Fix $\varepsilon > 0$ and choose $f_\varepsilon \in L^2(\mathbb{R})$ by $\hat{f}_\varepsilon(\xi) = \frac{1}{\sqrt{2\varepsilon}}\chi_{[\xi_0-\varepsilon,\xi_0+\varepsilon]}(\xi)$. Since $\|f_\varepsilon\|_2 = \|\hat{f}_\varepsilon\|_2 = 1$, using the fact that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B yields

$$A \le J_1(f_\varepsilon,M,\xi_0) + J_2(f_\varepsilon,M,\xi_0) \le B \quad \text{for all } M>0 \text{ and } \xi_0\in\mathbb{R}. \tag{5.4}$$

Remember that we chose $C_{\min} = \min_{\ell=1,\dots,L} C_\ell$ and $D_{\max} = \max_{\ell=1,\dots,L} D_\ell$. Obviously, $C_{\min}$ and $D_{\max}$ are both positive and finite.

First we derive estimates for $J_1(f_\varepsilon,M,\xi_0)$ and use these to prove the upper bound in (5.3). We claim that for any $M > 0$ and $\varepsilon \in (0, \frac{r}{M})$,

$$\begin{aligned} &\frac{C_{\min}}{2\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \sum_{\ell=1}^{L}\sum_{s\in S_\ell, s\le M} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2\,d\xi \\ &\le J_1(f_\varepsilon,M,\xi_0) \\ &\le \frac{D_{\max}}{2\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \sum_{\ell=1}^{L}\sum_{s\in S_\ell, s\le M} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2\,d\xi. \end{aligned} \tag{5.5}$$

To prove this, let $\ell \in \{1,\dots,L\}$. Since for each $s\in S_\ell$ with $s \le M$ and $k\in\mathbb{Z}$,

$$[\xi_0 + \tfrac{(2k-1)r}{s}, \xi_0 + \tfrac{(2k+1)r}{s}] \cap [\xi_0-\varepsilon,\xi_0+\varepsilon] \neq \emptyset \quad \iff \quad k=0,$$

we may rewrite $J_1(f_\varepsilon,M,\xi_0)$ as

$$\begin{aligned} &J_1(f_\varepsilon,M,\xi_0) \\ &= \sum_{\ell=1}^{L}\sum_{s\in S_\ell, s\le M}\sum_{t\in T_\ell} w_\ell(s)v_\ell(t)\frac{1}{s}\left|\int_{s\xi_0-r}^{s\xi_0+r} \hat{f}_\varepsilon(\tfrac{\xi}{s})\overline{\hat{\psi}_\ell(\xi)}e^{2\pi it\xi}\,d\xi\right|^2 \\ &= \sum_{\ell=1}^{L}\sum_{s\in S_\ell, s\le M} w_\ell(s)\frac{1}{s}\sum_{t\in T_\ell} v_\ell(t)\left|\int_{-r}^{r} \hat{f}_\varepsilon(\tfrac{\xi+s\xi_0}{s})\overline{\hat{\psi}_\ell(\xi+s\xi_0)}e^{2\pi it\xi}\,d\xi\right|^2 . \end{aligned}$$

Due to the definition of $C_{\min}$ and $D_{\max}$, for all $\ell = 1, \ldots, L$ the system $\mathcal{E}(T_\ell, v_\ell, r)$ is a frame for $L^2[-r, r]$ with frame bounds $C_{\min}$ and $D_{\max}$ for some $r > 0$. Hence

$$\begin{aligned} J_1(f_\varepsilon, M, \xi_0) &\leq D_{\max} \sum_{\ell=1}^{L} \sum_{s \in S_\ell, s \leq M} w_\ell(s) \frac{1}{s} \int_{-r}^{r} |\hat{f}_\varepsilon(\tfrac{\xi + s\xi_0}{s}) \hat{\psi}_\ell(\xi + s\xi_0)|^2 d\xi \\ &= D_{\max} \sum_{\ell=1}^{L} \sum_{s \in S_\ell, s \leq M} w_\ell(s) \int_{\xi_0 - \frac{r}{s}}^{\xi_0 + \frac{r}{s}} |\hat{f}_\varepsilon(\xi) \hat{\psi}_\ell(s\xi)|^2 d\xi \\ &= \frac{D_{\max}}{2\varepsilon} \int_{\xi_0 - \varepsilon}^{\xi_0 + \varepsilon} \sum_{\ell=1}^{L} \sum_{s \in S_\ell, s \leq M} w_\ell(s) |\hat{\psi}_\ell(s\xi)|^2 d\xi. \end{aligned}$$

This proves one part of (5.5). The other part can be treated similarly.

Now due to boundedness, we may let $\varepsilon \to 0$ and then $M \to \infty$ in (5.4) and (5.5), which yields

$$\sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi_0)|^2 \leq \frac{B}{C_{\min}} \quad \text{for a.e. } \xi_0 \in \mathbb{R}.$$

This proves the upper bound in (5.3).

Now let $\varepsilon = \frac{r}{2M}$. To prove the lower bound in (5.3), we study the second term $J_2(f_\varepsilon, M, \xi_0)$. First we show that for any $\xi_0 \in \mathbb{R} \backslash \{0\}$ and $\delta \in (0, A)$, there exists $M_0 > 0$ such that

$$J_2(f_\varepsilon, M, \xi_0) < \delta \quad \text{for all } M \geq M_0. \tag{5.6}$$

We will prove the claim only for $\xi_0 > 0$. The other case can be treated similarly. Since for each $k \in \mathbb{Z}$ with $|k| > \frac{s\varepsilon}{2r} + \frac{1}{2}$,

$$[\xi_0 + \tfrac{(2k-1)r}{s}, \xi_0 + \tfrac{(2k+1)r}{s}] \cap [\xi_0 - \varepsilon, \xi_0 + \varepsilon] = \emptyset,$$

we obtain the following estimate for each $\ell = 1, \ldots, L$ and $s \in S_\ell, s > M$. Again we will use the fact that $\mathcal{E}(T_\ell, v_\ell, r)$ is a frame for $L^2[-r, r]$ with frame bounds $C_{\min}$ and $D_{\max}$ for some $r > 0$. We further assume that $M \geq \max\{\frac{r}{\xi_0}, 1\}$. We compute

$$\begin{aligned} &\sum_{t \in T_\ell} v_\ell(t) \frac{1}{s} \left| \int_{s\xi_0 - r}^{s\xi_0 + r} \sum_{k \in \mathbb{Z}} \hat{f}(\tfrac{\xi + 2kr}{s}) \overline{\hat{\psi}_\ell(\xi + 2kr)} e^{2\pi i t(\xi + 2kr)} \, d\xi \right|^2 \\ &= \sum_{t \in T_\ell} v_\ell(t) \frac{1}{s} \left| \int_{s\xi_0 - r}^{s\xi_0 + r} \sum_{k \in \mathbb{Z}, |k| \leq \frac{s\varepsilon}{2r} + \frac{1}{2}} \hat{f}_\varepsilon(\tfrac{\xi + 2kr}{s}) \overline{\hat{\psi}_\ell(\xi + 2kr)} e^{2\pi i t(\xi + 2kr)} \, d\xi \right|^2 \\ &= \sum_{t \in T_\ell} v_\ell(t) \frac{1}{s} \left| \sum_{k \in \mathbb{Z}, |k| \leq \frac{s\varepsilon}{2r} + \frac{1}{2}} e^{4\pi i t k r} \int_{s\xi_0 - r}^{s\xi_0 + r} \hat{f}_\varepsilon(\tfrac{\xi + 2kr}{s}) \overline{\hat{\psi}_\ell(\xi + 2kr)} e^{2\pi i t \xi} \, d\xi \right|^2 \end{aligned}$$

$$\leq \sum_{t\in T_\ell} v_\ell(t)\frac{1}{s}\left(\frac{s\varepsilon}{r}+2\right)\sum_{k\in\mathbb{Z},|k|\leq\frac{s\varepsilon}{2r}+\frac{1}{2}}\left|\int_{s\xi_0-r}^{s\xi_0+r}\hat{f}_\varepsilon(\tfrac{\xi+2kr}{s})\overline{\hat{\psi}_\ell(\xi+2kr)}e^{2\pi it\xi}\,d\xi\right|^2$$

$$=\left(\frac{\varepsilon}{r}+\frac{2}{s}\right)\sum_{k\in\mathbb{Z},|k|\leq\frac{s\varepsilon}{2r}+\frac{1}{2}}\sum_{t\in T_\ell} v_\ell(t)\left|\int_{-r}^{r}\hat{f}_\varepsilon(\tfrac{\xi+s\xi_0+2kr}{s})\overline{\hat{\psi}_\ell(\xi+s\xi_0+2kr)}\right.$$

$$\left.\cdot e^{2\pi it\xi}\,d\xi\right|^2$$

$$\leq D_{\max}\left(\frac{\varepsilon}{r}+\frac{2}{s}\right)\sum_{k\in\mathbb{Z},|k|\leq\frac{s\varepsilon}{2r}+\frac{1}{2}}\int_{-r}^{r}|\hat{f}_\varepsilon(\tfrac{\xi+s\xi_0+2kr}{s})\hat{\psi}_\ell(\xi+s\xi_0+2kr)|^2d\xi$$

$$\leq D_{\max}\frac{1}{2\varepsilon}\left(\frac{\varepsilon}{r}+\frac{2}{s}\right)\sum_{k\in\mathbb{Z},|k|\leq\frac{s\varepsilon}{2r}+\frac{1}{2}}\int_{-r}^{r}|\hat{\psi}_\ell(\xi+s\xi_0+2kr)|^2d\xi$$

$$\leq D_{\max}\left(\frac{1}{2r}+\frac{1}{s\varepsilon}\right)\int_{s(\xi_0-\varepsilon)-2r}^{s(\xi_0+\varepsilon)+2r}|\hat{\psi}_\ell(\xi)|^2d\xi$$

$$\leq \frac{5D_{\max}}{2r}\int_{s\frac{\xi_0}{2}-2r}^{3s\frac{\xi_0}{2}+2r}|\hat{\psi}_\ell(\xi)|^2d\xi.$$

The last inequality follows from

$$s(\xi_0+\varepsilon)+2r = s\xi_0+\frac{sr}{2M}+2r \leq s\xi_0+\frac{sr\xi_0}{2r}+2r = 3s\frac{\xi_0}{2}+2r,$$

a similar computation to show that $s(\xi_0-\varepsilon)-2r \geq s\frac{\xi_0}{2}-2r$, and the fact that $\frac{1}{s\varepsilon}<\frac{1}{M\varepsilon}=\frac{2}{r}$. Therefore, we may estimate $J_2(f_\varepsilon, M, \xi_0)$ in the following way:

$$J_2(f_\varepsilon, M, \xi_0) \leq \frac{5D_{\max}}{2r}\sum_{\ell=1}^{L}\sum_{s\in S_\ell, s>M} w_\ell(s)\int_{s\frac{\xi_0}{2}-2r}^{3s\frac{\xi_0}{2}+2r}|\hat{\psi}_\ell(\xi)|^2d\xi. \tag{5.7}$$

Fix $\ell\in\{1,\ldots,L\}$. It remains to prove that the right-hand-side for this particular ℓ converges to 0 as $M\to\infty$. First let $h>\max\{2\ln 4, |\ln(2r)|\}$ and notice that the sequence of intervals $\{e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})\}_{j\in\mathbb{Z}}$ is a tiling of $\mathbb{R}^+$. Moreover, we let M be large enough that $\ln(\frac{M\xi_0}{2}-2r)\geq 3h$, and define

$$j_M := \lfloor \tfrac{1}{h}\ln(\tfrac{M\xi_0}{2}-2r)-\tfrac{1}{2}\rfloor.$$

It is easy to check that for any $s\in S_\ell$, $s>M$,

$$[s\tfrac{\xi_0}{2}-2r, 3s\tfrac{\xi_0}{2}+2r]\cap e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})\neq\emptyset \implies j\geq j_M.$$

We further obtain that this intersection is nonempty if and only if

$$s\tfrac{\xi_0}{2}-2r < e^{jh+\frac{h}{2}} \quad\text{and}\quad 3s\tfrac{\xi_0}{2}+2r\geq e^{jh-\frac{h}{2}},$$

which implies

$$s \in [\tfrac{2}{3\xi_0}(e^{jh-\frac{h}{2}} - 2r), \tfrac{2}{\xi_0}(e^{jh+\frac{h}{2}} + 2r)).$$

Now we intend to show that this interval is contained in some $x[e^{-h}, e^h)$ with $x \in \mathbb{R}^+$ suitably chosen. Let $j \in \mathbb{Z}$ with $j \geq j_M$. Then, by the choice of h,

$$\begin{aligned}
[\tfrac{2}{3\xi_0}(e^{jh-\frac{h}{2}} - 2r), \tfrac{2}{\xi_0}(e^{jh+\frac{h}{2}} + 2r)) &= \tfrac{2}{\xi_0}e^{jh}[\tfrac{1}{3}(e^{-\frac{h}{2}} - 2re^{-jh}), e^{\frac{h}{2}} + 2re^{-jh}) \\
&\subseteq \tfrac{2}{\xi_0}e^{jh}[\tfrac{1}{3}(e^{-\frac{h}{2}} - 2re^{-j_M h}), e^{\frac{h}{2}} + 2re^{-j_M h}) \\
&\subseteq \tfrac{2}{\xi_0}e^{jh}[\tfrac{1}{3}(e^{-\frac{h}{2}} - e^{-h}), e^{\frac{h}{2}} + e^{-h}) \\
&\subseteq \tfrac{2}{\xi_0}e^{jh}[e^{-h}, e^h).
\end{aligned}$$

By Propositions 5.2(i) and 4.17, we have $\mathcal{D}^+(S_\ell, w_\ell) < \infty$. Then Proposition 4.15 implies that there exists $N < \infty$ such that

$$\begin{aligned}
&\#_{w_\ell}\{s \in S_\ell, s > M : [s\tfrac{\xi_0}{2} - 2r, 3s\tfrac{\xi_0}{2} + 2r] \cap e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}}) \neq \emptyset\} \\
&\qquad \leq \#_{w_\ell}\{S_\ell \cap \tfrac{2}{\xi_0}e^{jh}[e^{-h}, e^h)\} \leq N
\end{aligned}$$

for any $j \in \mathbb{Z}$, $j \geq j_M$. Thus,

$$\begin{aligned}
&\sum_{s\in S_\ell, s>M} w_\ell(s) \int_{s\frac{\xi_0}{2}-2r}^{3s\frac{\xi_0}{2}+2r} |\hat{\psi}_\ell(\xi)|^2 d\xi \\
&= \sum_{s\in S_\ell, s>M} w_\ell(s) \sum_{j\in\mathbb{Z}} \int_{[s\frac{\xi_0}{2}-2r, 3s\frac{\xi_0}{2}+2r]\cap e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} |\hat{\psi}_\ell(\xi)|^2 d\xi \\
&= \sum_{j\in\mathbb{Z}, j\geq j_M} \sum_{s\in S_\ell, s>M} w_\ell(s) \int_{[s\frac{\xi_0}{2}-2r, 3s\frac{\xi_0}{2}+2r]\cap e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} |\hat{\psi}_\ell(\xi)|^2 d\xi \\
&\leq \sum_{j\in\mathbb{Z}, j\geq j_M} N \int_{e^{jh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})} |\hat{\psi}(\xi)|^2 d\xi,
\end{aligned}$$

which converges to 0 as $M \to \infty$. Together with (5.7) this proves (5.6).

Now (5.4), (5.5), and (5.6) imply that with $\varepsilon = \frac{r}{2M}$, for $\xi_0 \in \mathbb{R}\backslash\{0\}$ and $\delta \in (0, A)$, we have

$$A - \delta \leq J_1(f_\varepsilon, M, \xi_0) \leq \frac{D_{\max}}{2\varepsilon} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2 \, d\xi,$$

provided M is large enough. Letting $M \to \infty$ yields

$$A - \delta \leq D_{\max} \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi_0)|^2 \quad \text{for a.e. } \xi_0 \in \mathbb{R}.$$

Since δ was arbitrary, finally we obtain

$$\frac{A}{D_{\max}} \leq \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi_0)|^2 \quad \text{for a.e. } \xi_0 \in \mathbb{R}.$$

This proves the lower bound in (5.3). □

Now we will show that a Littlewood–Paley type inequality yields a relationship between the density of the associated sequences, the bounds, and special constants depending on the considered functions. The decomposition technique employed in the proof is inspired by a similar technique used by Kolountzakis and Lagarias [91].

Proposition 5.4. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given such that $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$. Further, let $f_1, \ldots, f_L \in L^1(\mathbb{R})\backslash\{0\}$ with $f_\ell \geq 0$ and $\int_0^\infty \frac{f_\ell(x)}{x}\, dx < \infty$ for $\ell = 1, \ldots, L$. Suppose that*

$$A \leq \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) f_\ell(sx) \leq B \quad \text{for a.e. } x \in \mathbb{R}^+. \tag{5.8}$$

Then

$$A \leq \mathcal{D}^-(\{(S_\ell, w_\ell \cdot \int_0^\infty \frac{f_\ell(x)}{x}\, dx)\}_{\ell=1}^L) \leq \mathcal{D}^+(\{(S_\ell, w_\ell \cdot \int_0^\infty \frac{f_\ell(x)}{x}\, dx)\}_{\ell=1}^L) \leq B.$$

Proof. For the sake of brevity, in this proof for each $h > 0$ we will define $K_h \subseteq \mathbb{R}^+$ by $K_h = [e^{-\frac{h}{2}}, e^{\frac{h}{2}})$. Let $\varepsilon > 0$. Since $\int_0^\infty \frac{f_\ell(x)}{x}\, dx < \infty$, we can choose some $c > 0$ such that $\int_{\mathbb{R}^+ \backslash K_c} \frac{f_\ell(x)}{x}\, dx < \varepsilon$ for all $\ell = 1, \ldots, L$. Further, fix $y \in \mathbb{R}^+$ and $h > c$. Dividing inequality (5.8) by x and integrating each term over the box $y^{-1}K_h$ yields

$$hA \leq \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \int_{y^{-1}K_h} \frac{f_\ell(sx)}{x}\, dx \leq hB.$$

Then we make the decomposition

$$\sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \int_{y^{-1}K_h} \frac{f_\ell(sx)}{x}\, dx = I_1(y,h) - I_2(y,h) + I_3(y,h) + I_4(y,h),$$

where

$$I_1(y,h) = \sum_{\ell=1}^{L} \sum_{s \in S_\ell \cap yK_{h-c}} w_\ell(s) \int_0^\infty \frac{f_\ell(sx)}{x}\, dx,$$

$$I_2(y,h) = \sum_{\ell=1}^{L} \sum_{s \in S_\ell \cap yK_{h-c}} w_\ell(s) \int_{\mathbb{R}^+ \backslash y^{-1}K_h} \frac{f_\ell(sx)}{x}\, dx,$$

$$I_3(y,h) = \sum_{\ell=1}^{L} \sum_{s\in S_\ell\cap(yK_{h+c}\backslash yK_{h-c})} w_\ell(s) \int_{y^{-1}K_h} \frac{f_\ell(sx)}{x}\,dx,$$

$$I_4(y,h) = \sum_{\ell=1}^{L} \sum_{s\in S_\ell\cap(\mathbb{R}^+\backslash yK_{h+c})} w_\ell(s) \int_{y^{-1}K_h} \frac{f_\ell(sx)}{x}\,dx.$$

By Proposition 4.17, we have $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ for each $\ell = 1, \ldots, L$. By Proposition 4.15, there exists $N < \infty$ such that

$$\#_{w_\ell}(S_\ell \cap xK_t) \le (t+1) \sup_{\tilde{x}\in\mathbb{R}^+} \#_{w_\ell}(S_\ell \cap \tilde{x}K_1) \le (t+1)N$$

for all $x \in \mathbb{R}^+$, $t > 0$, and $\ell = 1, \ldots, L$.

We first observe that

$$I_1(y,h) = \sum_{\ell=1}^{L} \#_{w_\ell}(S_\ell \cap yK_{h-c}) \int_0^\infty \frac{f_\ell(x)}{x}\,dx.$$

To estimate $I_2(y,h)$, note that if $s \in yK_{h-c}$, then we have $s(\mathbb{R}^+\backslash y^{-1}K_h) = \mathbb{R}^+\backslash sy^{-1}K_h \subseteq \mathbb{R}^+\backslash K_c$. Therefore the contribution of $I_2(y,h)$ can be controlled by

$$\begin{aligned} I_2(y,h) &\le \sum_{\ell=1}^{L} \sum_{s\in S_\ell\cap yK_{h-c}} w_\ell(s) \int_{\mathbb{R}^+\backslash K_c} \frac{f_\ell(x)}{x}\,dx \\ &\le \sum_{\ell=1}^{L} \#_{w_\ell}(S_\ell \cap yK_{h-c})\,\varepsilon \\ &\le L(h-c+1)N\varepsilon. \end{aligned}$$

Since $K_{h+c}\backslash K_{h-c}$ can be covered by a union of at most $2c+1$ intervals of the form xK_1, $x \in \mathbb{R}^+$, the term $I_3(y,h)$ can be estimated as follows:

$$\begin{aligned} I_3(y,h) &\le \sum_{\ell=1}^{L} \sum_{s\in S_\ell\cap(yK_{h+c}\backslash yK_{h-c})} w_\ell(s) \int_0^\infty \frac{f_\ell(x)}{x}\,dx \\ &= \sum_{\ell=1}^{L} \#_{w_\ell}(S_\ell \cap y(K_{h+c}\backslash K_{h-c})) \int_0^\infty \frac{f_\ell(x)}{x}\,dx \\ &\le (2c+1)N \sum_{\ell=1}^{L} \int_0^\infty \frac{f_\ell(x)}{x}\,dx. \end{aligned}$$

To estimate $I_4(y,h)$, note that if $s \notin yK_{h+c}$, then $sy^{-1}K_h \subseteq \mathbb{R}^+\backslash K_c$. Furthermore, each interval in $\{sy^{-1}K_h : s \in S\}$ can intersect at most $h+1$ of the other intervals in the set. Hence the contribution of $I_4(y,h)$ can be controlled by

$$\begin{aligned} I_4(y,h) &= \sum_{\ell=1}^{L} \sum_{s \in S_\ell \cap (\mathbb{R}^+ \setminus y K_{h+c})} w_\ell(s) \int_{s y^{-1} K_h} \frac{f_\ell(x)}{x}\, dx \\ &\le L(h+1)N \int_{\mathbb{R}^+ \setminus K_c} \frac{f_\ell(x)}{x}\, dx \\ &\le L(h+1)N\varepsilon. \end{aligned}$$

Combining these estimates, we see that

$$\begin{aligned} hA \le \sum_{\ell=1}^{L} \#_{w_\ell}(S_\ell \cap y K_{h-c}) \int_0^\infty \frac{f_\ell(x)}{x}\, dx + L(h-c+1)N\varepsilon \\ +(2c+1)N \sum_{\ell=1}^{L} \int_0^\infty \frac{f_\ell(x)}{x}\, dx + L(h+1)N\varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} A &= \liminf_{h\to\infty} \frac{hA}{h} \\ &\le \liminf_{h\to\infty} \inf_{y\in\mathbb{R}^+} \frac{\sum_{\ell=1}^{L} \#_{w_\ell}(S_\ell \cap y K_{h-c}) \int_0^\infty \frac{f_\ell(x)}{x}\, dx}{h} \\ &\quad + \limsup_{h\to\infty} \frac{L(h-c+1)N\varepsilon}{h} + \limsup_{h\to\infty} \frac{(2c+1)N}{h} \sum_{\ell=1}^{L} \int_0^\infty \frac{f_\ell(x)}{x}\, dx \\ &\quad + \limsup_{h\to\infty} \frac{L(h+1)N\varepsilon}{h} \\ &= \mathcal{D}^-(\{(S_\ell, w_\ell \cdot \int_0^\infty \frac{f_\ell(x)}{x}\, dx)\}_{\ell=1}^{L}) + 2LN\varepsilon. \end{aligned}$$

Now letting ε go to zero yields $A \le \mathcal{D}^-(\{(S_\ell, w_\ell \cdot \int_0^\infty \frac{f_\ell(x)}{x}\, dx)\}_{\ell=1}^{L})$. The second claim, $\mathcal{D}^+(\{(S_\ell, w_\ell \cdot \int_0^\infty \frac{f_\ell(x)}{x}\, dx)\}_{\ell=1}^{L}) \le B$, can be treated similarly. Hence the theorem is proved. □

Next we derive a relationship between the density of the sequence of time indices and the frame bounds of a frame of weighted exponentials. This result contains a result by Heil and Kutyniok [74] as a special case.

Theorem 5.5. *Let $T_1, \ldots, T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given such that $\bigcup_{\ell=1}^{L} \mathcal{E}(T_\ell, v_\ell, r)$ is a frame for $L^2[-r,r]$ with frame bounds A and B for some $r > 0$. Then the following statements hold.*

(i) *We have*

$$A \le \mathcal{D}^-(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) \le \mathcal{D}^+(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) \le B.$$

(ii) *If the system* $\bigcup_{\ell=1}^{L} \mathcal{E}(T_\ell, v_\ell, r)$ *is a tight frame for* $L^2[-r,r]$ *with frame bound* A*, then* $\{(T_\ell, v_\ell)\}_{\ell=1}^{L}$ *has uniform weighted Beurling density and*

$$\mathcal{D}(\{(T_\ell, v_\ell)\}_{\ell=1}^{L}) = A.$$

(iii) *If* $T_1 = \ldots = T_L = T$*,* $v_1 = \ldots = v_L = 1$*, and* $\mathcal{E}(T,r)$ *is a tight frame for* $L^2[-r,r]$ *with frame bound* A*, then* T *has uniform Beurling density and*

$$\mathcal{D}(T) = A.$$

Proof. Only part (i) will be proven. Parts (ii) and (iii) then follow as corollaries.

For the proof of (i) we will employ Proposition 5.4. For $\xi \in \mathbb{R}$, consider the function $g_\xi(x) = M_\xi \chi_{[-r,r]}(x) = e^{2\pi i \xi x} \chi_{[-r,r]}(x)$. By definition of frame,

$$A \left\| g_\xi \right\|_2^2 \le \sum_{\ell=1}^{L} \sum_{t \in T_\ell} v_\ell(t) |\left\langle g_\xi, M_t \chi_{[-r,r]} \right\rangle|^2 \le B \left\| g_\xi \right\|_2^2 .$$

We compute

$$\left|\left\langle g_\xi, M_t \chi_{[-r,r]} \right\rangle\right|^2 = \begin{cases} 4r^2, & \xi = t, \\ \frac{\sin^2(2\pi(\xi - t)r)}{\pi^2(\xi - t)^2}, & \xi \neq t, \end{cases}$$

and define $h \in L^1(\mathbb{R})$ by

$$h(x) = \begin{cases} 4r^2, & x = 0, \\ \frac{\sin^2(2\pi x r)}{\pi^2 x^2}, & x \neq 0. \end{cases}$$

Using, in addition, the fact that $\left\| g_\xi \right\|_2^2 = 2r$, we obtain

$$2rA \le \sum_{\ell=1}^{L} \sum_{t \in T_\ell} v_\ell(t) h(t - \xi) \le 2rB \quad \text{for all } \xi \in \mathbb{R}.$$

Now define $f \in L^1(\mathbb{R}^+)$ by

$$f(x) = h(\ln(x)),$$

and define $\tilde{v}_\ell : e^{T_\ell} \to \mathbb{R}^+$ by $\tilde{v}_\ell(e^t) = v_\ell(t)$. Then $h(t - \xi) = f(e^{t-\xi}) = f(e^t e^{-\xi})$, and hence

$$2rA \le \sum_{\ell=1}^{L} \sum_{t \in T_\ell} \tilde{v}_\ell(e^t) f(e^t x) \le 2rB \quad \text{for all } x \in \mathbb{R}^+.$$

Further, an easy calculation shows that

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_{-\infty}^\infty \frac{\sin^2(2\pi xr)}{\pi^2 x^2}\,dx = 2r.$$

By Proposition 5.2(ii), we have $\mathcal{D}^+(\{(T_\ell, v_\ell)\}_{\ell=1}^L) < \infty$. Observing that for $x \in \mathbb{R}$,

$$\#_{v_\ell}(T_\ell \cap x + [-\tfrac{h}{2}, \tfrac{h}{2}]) = \#_{\tilde{v}_\ell}(e^{T_\ell} \cap e^x[e^{-\frac{h}{2}}, e^{\frac{h}{2}}])$$

and using the definition of density for sequences in $\mathbb{R}^+$ and in $\mathbb{R}$, yields $\mathcal{D}^+(\{(e^{T_\ell}, \tilde{v}_\ell)\}_{\ell=1}^L) < \infty$. Now Proposition 5.4 implies that

$$2rA \le \mathcal{D}^-(\{(e^{T_\ell}, \tilde{v}_\ell \cdot 2r)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(e^{T_\ell}, \tilde{v}_\ell \cdot 2r)\}_{\ell=1}^L) \le 2rB.$$

Using the same argument concerning the relation between the density of sequences in $\mathbb{R}^+$ and $\mathbb{R}$ as above again and dividing by $2r$ proves the claim. □

The following result establishes a fundamental relationship between weighted affine density, the frame bounds, and the admissibility constants for weighted wavelet frames with finitely many generators. This includes the result by Chui and Shi and by Daubechies (5.1) (see Remark 5.7(b)).

Theorem 5.6. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$, and $T_1, \ldots, T_L \subseteq \mathbb{R}$ with associated weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Further, let $\psi_1, \ldots, \psi_L \in L^2_A(\mathbb{R})\backslash\{0\}$. If $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B, and if for all $\ell = 1, \ldots, L$ the system $\mathcal{E}(T_\ell, v_\ell, r)$ is a frame for $L^2[-r, r]$ with frame bounds C_ℓ and D_ℓ for some $r > 0$, then the following statements hold, where we set $C_{\min} = \min_{\ell=1,\ldots,L} C_\ell$ and $D_{\max} = \max_{\ell=1,\ldots,L} D_\ell$.*

(i) *We have*

$$\frac{A}{D_{\max}} \le \mathcal{D}^-(\{(S_\ell, C^+_{\psi_\ell} w_\ell)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(S_\ell, C^+_{\psi_\ell} w_\ell)\}_{\ell=1}^L) \le \frac{B}{C_{\min}} \tag{5.9}$$

and

$$\frac{A}{D_{\max}} \le \mathcal{D}^-(\{(S_\ell, C^-_{\psi_\ell} w_\ell)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(S_\ell, C^-_{\psi_\ell} w_\ell)\}_{\ell=1}^L) \le \frac{B}{C_{\min}}. \tag{5.10}$$

(ii) *Suppose $\mathcal{E}(T_\ell, v_\ell, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$ with the same frame bound $C_{\min} = D_{\max}$. Then*

$$\begin{aligned} A \le \mathcal{D}^-(\{(S_\ell \times T_\ell, (C^+_{\psi_\ell} w_\ell, v_\ell))\}_{\ell=1}^L) \\ \le \mathcal{D}^+(\{(S_\ell \times T_\ell, (C^+_{\psi_\ell} w_\ell, v_\ell))\}_{\ell=1}^L) \le B \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} A \le \mathcal{D}^-(\{(S_\ell \times T_\ell, (C^-_{\psi_\ell} w_\ell, v_\ell))\}_{\ell=1}^L) \\ \le \mathcal{D}^+(\{(S_\ell \times T_\ell, (C^-_{\psi_\ell} w_\ell, v_\ell))\}_{\ell=1}^L) \le B \end{aligned} \tag{5.12}$$

(iii) *Suppose* $\mathcal{E}(T_\ell, v_\ell, r)$ *is a tight frame for* $L^2[-r,r]$ *for some* $r > 0$ *with the same frame bound* $C_{\min} = D_{\max}$*, and we have* $w_\ell = v_\ell = 1$ *and* $C_{\psi_\ell} = C$ *for all* $\ell = 1, \dots, L$*. Then*

$$A \leq \frac{1}{2}\,\mathcal{D}^-(\bigcup_{\ell=1}^{L}(S_\ell \times T_\ell)) \cdot C \leq \frac{1}{2}\,\mathcal{D}^+(\bigcup_{\ell=1}^{L}(S_\ell \times T_\ell)) \cdot C \leq B.$$

Proof. Inequality (5.9) follows immediately from Propositions 5.2(i), 5.3 and 5.4. Also inequality (5.10) is implied by those three propositions, only here we have to apply Proposition 5.4 to $\xi \mapsto |\hat{\psi}_\ell(-\xi)|^2$ for $\ell = 1, \dots, L$ instead.

Further notice that (iii) is an immediate consequence of (ii).

Thus it remains to prove part (ii). Suppose that for all $\ell = 1, \dots, L$ the system $\mathcal{E}(T_\ell, v_\ell, r)$ is a tight frame for $L^2[-r,r]$ for some $r > 0$ with frame bound D. By Theorem 5.5(ii), each pair (T_ℓ, v_ℓ) has uniform weighted Beurling density $\mathcal{D}(T_\ell, v_\ell) = D$. Then Lemma 5.1 implies that

$$\mathcal{D}^-(\{(S_\ell, C^+_{\psi_\ell} w_\ell)\}_{\ell=1}^L) = \frac{\mathcal{D}^-(\{(S_\ell \times T_\ell, (C^+_{\psi_\ell} w_\ell, v_\ell))\}_{\ell=1}^L)}{D}$$

and a similar result holds for the upper density. Since $D = D_{\max} = C_{\min}$, this shows that (5.11) and (5.12) follow from (5.9) and (5.10). □

Remark 5.7. (a) In general the hypothesis that $\mathcal{E}(T, r)$ is a frame for $L^2[-r,r]$ for some $r > 0$ is not restrictive, since it was shown by Jaffard in [85, Lem. 2] that $\mathcal{E}(T, r)$ is a frame for $L^2[-r,r]$ for some $r > 0$ if and only if T is the disjoint union of a sequence with a uniform density and a finite number of uniformly discrete sequences, i.e., of sequences Δ which satisfy $\inf_{t_1,t_2 \in \Delta, t_1 \neq t_2} |t_1 - t_2| > 0$. This is easily seen to be equivalent to $0 < \mathcal{D}^-(T) \leq \mathcal{D}^+(T) < \infty$ (see, for instance, Heil and Kutyniok [74]).

However, $\mathcal{W}(\psi, \Lambda)$ being a frame for $L^2(\mathbb{R})$ does not imply $\mathcal{E}(T, r)$ being a frame for $L^2[-r,r]$ for some $r > 0$. A counterexample for this fact was derived by Sun and Zhou in [121, Ex. 2.1].

(b) Consider the case $S = \{a^j\}_{j \in \mathbb{Z}}$, $a > 1$ and $T = b\mathbb{Z}$, $b > 0$. Then Lemma 3.3 shows that $\mathcal{D}^-(S \times T) = \mathcal{D}^+(S \times T) = \frac{1}{b \ln a}$. Therefore Theorem 5.6(iii) contains (5.1) as a special case.

Theorem 5.6 yields several results interesting in their own right, which are all direct implications of this theorem. Here we focus on the singly generated situation.

It is conjectured that $\mathcal{W}(\psi, \Lambda)$ being a frame for $L^2(\mathbb{R})$ implies $\mathcal{D}^-(\Lambda) > 0$, most likely under mild conditions on ψ or Λ. In this book we obtained one partial result in Theorem 4.2 and will derive another in Corollary 6.12. The general conjecture is still unsolved even in the nonweighted case. For further partial results we refer to Sun [117]. With the following corollary we prove yet another special case of this conjecture. This result is in fact a generalization of one part of a result from Sun and Zhou [121, Thm. 2.1] and Sun [117, Thm. 1.1] to the weighted situation.

Corollary 5.8. *Let* $\Lambda = S \times T \subseteq \mathbb{A}$*,* $w : S \to \mathbb{R}^+$*, and* $v : T \to \mathbb{R}^+$*, and let* $\psi \in L^2_A(\mathbb{R})$*. If* $\mathcal{W}(\psi, \Lambda, (w, v))$ *is a frame for* $L^2(\mathbb{R})$*, and* $\mathcal{E}(T, v, r)$ *is a tight frame for* $L^2[-r, r]$ *for some* $r > 0$*, then* $\mathcal{D}^-(\Lambda, (w, v)) > 0$*.*

The second corollary shows that a wavelet system can only form a tight frame provided that the associated sequence of time-scale indices possesses a uniform affine density, thereby also delivering the exact value of the frame bound in terms of the affine density of the sequence of time-scale indices and the admissibility constant of the analyzing wavelet. This should be compared with the fact that the sequence of time-scale indices of classical affine systems always possesses a uniform affine density (Lemma 3.3). The following result moreover provides one reason, why there does not exist a Nyquist phenomenon for wavelet systems (see Section 5.4).

Corollary 5.9. *Let* $\Lambda = S \times T \subseteq \mathbb{A}$*,* $w : S \to \mathbb{R}^+$*, and* $v : T \to \mathbb{R}^+$*, and let* $\psi \in L^2_A(\mathbb{R})$*. If* $\mathcal{W}(\psi, \Lambda, (w, v))$ *is a tight frame for* $L^2(\mathbb{R})$ *with frame bound* A*, and* $\mathcal{E}(T, v, r)$ *is a tight frame for* $L^2[-r, r]$ *for some* $r > 0$*, then* $(\Lambda, (w, v))$ *has uniform weighted affine density and*

$$A = \mathcal{D}(\Lambda, (w, v)) \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi = \mathcal{D}(\Lambda, (w, v)) \int_{-\infty}^0 \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi. \tag{5.13}$$

5.4 The Nyquist Phenomenon

In this section we discuss the impact of Corollary 5.9 on the existence of a Nyquist density for wavelet systems. To present our observations in a clear way, we restrict ourselves to singly generated nonweighted wavelet systems. However, we remark that these considerations can be extended to weighted wavelet systems with finitely many generators.

In brief, in terms of necessary conditions for Gabor frames there is a critical or Nyquist density for Λ separating frames from non-frames, and furthermore the Riesz bases sit exactly at this critical density. It is natural to ask whether wavelet systems share similar properties, and the immediate answer is that there is clearly no exact analogue of the Nyquist density for analyzing wavelets. In particular, consider the case of the classical affine systems $\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k\in\mathbb{Z}})$ with dilation parameter $a > 1$ and translation parameter $b > 0$. It can be shown that for *each* $a > 1$ and $b > 0$ there exists an analyzing wavelet $\psi \in L^2(\mathbb{R})$ such that $\mathcal{W}(\psi, \Lambda)$ is a frame or even an orthonormal basis for $L^2(\mathbb{R})$ (see Dai and Larson [38, Ex. 4.5, Part 10]). In fact, the wavelet set construction of Dai, Larson, and Speegle [39] shows that this is true even in higher dimensions: wavelet orthonormal bases in the classical affine form exist for any expansive dilation matrix. For additional demonstrations of the impossibility of a Nyquist density, even given constraints on the norm or on the admissibility constant of the analyzing wavelet, see the example of Daubechies in [40, Thm. 2.10] and the more extensive analysis of Balan in [5].

We will now demonstrate how the density relation (5.13) reveals one reason why there does not exist a Nyquist density for wavelet orthonormal bases, whereas there exists one for Gabor orthonormal bases. We can view the necessary density condition (5.13) for $\mathcal{W}(\psi, \Lambda)$ to be a Parseval frame for $L^2(\mathbb{R})$, where $\mathcal{E}(T, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$, also from the following perspective:

$$\mathcal{D}(\Lambda) = 2\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\xi}{|\xi|})}. \tag{5.14}$$

Thus once $\mathcal{W}(\psi, \Lambda)$ constitutes an orthonormal basis for $L^2(\mathbb{R})$, and $\mathcal{E}(T, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$, we obtain (5.14) as a necessary condition. The wavelet system being an orthonormal basis implies $\|\hat{\psi}\|_2 = 1$. However, we do not have any control over the constant $\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\xi}{|\xi|})}$. Thus although Λ has a uniform affine density in this case, the value of it can range over the whole positive axis. As mentioned above, it can be shown that for each dilation parameter $a > 1$, there exists an analyzing wavelet $\psi \in L^2(\mathbb{R})$ such that $\mathcal{W}(\psi, \{(a^j, k)\}_{j,k\in\mathbb{Z}})$ is an orthonormal basis for $L^2(\mathbb{R})$. Since $\mathcal{D}(\{(a^j, k)\}_{j,k\in\mathbb{Z}}) = \frac{1}{\ln a}$ by Lemma 3.3, the affine density can attain each positive value. Thus Corollary 5.9 reveals one reason, why wavelet systems do not possess a Nyquist density.

This consideration should be compared to Theorem 7.19, which shows in particular, that if a Gabor system $\mathcal{G}(g, \Lambda)$, where $g \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$, forms an orthonormal basis for $L^2(\mathbb{R})$, it has to satisfy

$$\mathcal{D}(\Lambda) = \|\hat{g}\|_2^{-2}.$$

In this situation $\|\hat{g}\|_2 = 1$ immediately implies $\mathcal{D}(\Lambda) = 1$ in contrast to the wavelet systems, for which the norm of $\hat{\psi}$ needed for the computation of the uniform density is equipped with a different measure.

5.5 Sufficient Density Conditions for Wavelet Frames

Up to now density conditions have only served as necessary conditions. In this section we now show that density conditions can in fact be used to characterize the existence of weighted wavelet frames. To prove this result we need the following technical lemma.

Lemma 5.10. *Let $S \subseteq \mathbb{R}^+$ and $w : S \to \mathbb{R}^+$, and let f be in $L^1(\mathbb{R}) \cap C(\mathbb{R})$ with $f \geq 0$ and $f(x) \leq a|x|^\alpha$ as $|x| \to 0$ for some $a, \alpha > 0$. If $\mathcal{D}^+(S, w) < \infty$, then for each $\varepsilon > 0$ there exists $\gamma \in (0, 1)$ such that*

$$\sum_{s\in S} w(s) f(sx) \chi_{[0,\gamma)}(s|x|) < \varepsilon \quad \textit{for all } x \in \mathbb{R}.$$

Proof. Fix $\varepsilon > 0$, and let $\nu \in (0,1)$ be chosen so that $f(x) \le a|x|^\alpha$ for all $|x| \le \nu$. Since $\mathcal{D}^+(S,w) < \infty$, Proposition 4.15 shows the existence of some $N < \infty$ such that $\#_w(S \cap x[\nu,1)) \le N$ for all $x \in \mathbb{R}^+$. Then, for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{s\in S} w(s) f(sx)\chi_{[0,\nu^n)}(s|x|) &\le a \sum_{s\in S\cap[0,|x|^{-1}\nu^n)} w(s)\,(s|x|)^\alpha \\ &= a|x|^\alpha \sum_{j=n}^{\infty} \sum_{s\in S\cap|x|^{-1}[\nu^{j+1},\nu^j)} w(s)\, s^\alpha \\ &\le a|x|^\alpha \sum_{j=n}^{\infty} N(|x|^{-1}\nu^j)^\alpha \\ &= aN \sum_{j=n}^{\infty} (\nu^\alpha)^j,\end{aligned}$$

which is finite. Thus we can choose $n_0 \in \mathbb{N}$ such that

$$\sum_{s\in S} w(s) f(sx)\chi_{[0,\nu^{n_0})}(s|x|) < \varepsilon.$$

Setting $\gamma = \nu^{n_0}$ settles the claim. □

The next result shows that the existence of frames of band-limited admissible wavelets with a certain decay condition can be characterized by using a condition on the density of the sequences of scale indices. This result is inspired by a result by Yang and Zhou [127, Cor. 1] for nonweighted singly generated wavelet systems, in which our density condition (ii) was substituted by a Littlewood–Paley type condition on the sequence of scale indices.

Theorem 5.11. *Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Further, let $\psi_1, \ldots, \psi_L \in L^1(\mathbb{R}) \cap L^2_A(\mathbb{R})$ with $|\hat{\psi}_\ell(\xi)| \le a|\xi|^\alpha$ as $|\xi| \to 0$ for some $a, \alpha > 0$, where $\xi = 0$ is an isolated zero of $\hat{\psi}_\ell$, and $|\hat{\psi}_\ell(\xi)| = 0$ for any $|\xi| \ge \Omega$ for all $\ell = 1, \ldots, L$. Then the following conditions are equivalent.*

(i) *There exists $T \subseteq \mathbb{R}$ such that $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$, where $r > 2\Omega$, and $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times T, (w_\ell, 1))$ is a frame for $L^2(\mathbb{R})$.*
(ii) $0 < \mathcal{D}^-(\{(S_\ell, w_\ell)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$.

Moreover, if (ii) holds, then $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ is a frame for $L^2(\mathbb{R})$ for any $T_\ell \subseteq \mathbb{R}$ equipped with weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ satisfying that $\mathcal{E}(T_\ell, v_\ell, r)$ constitutes a frame for $L^2[-r,r]$ for all $\ell = 1, \ldots, L$, where $r > 2\Omega$.

Proof. The implication (i) ⇒ (ii) follows immediately from Theorem 5.6.

Now suppose (ii) holds. First we show that (i) and the moreover-part follow immediately from the existence of $0 < A \le B < \infty$ such that

$$A \le \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi)|^2 \le B \quad \text{for all } \xi \in \mathbb{R}. \tag{5.15}$$

For this, let $T_1, \ldots, T_L$ be sequences in $\mathbb{R}$ equipped with weight functions $v_\ell : T_\ell \to \mathbb{R}^+$ satisfying that $\mathcal{E}(T_\ell, v_\ell, r)$ constitutes a frame for $L^2[-r,r]$ for all $\ell = 1, \ldots, L$, where $r > 2\Omega$. We further suppose that (5.15) holds. Let $f \in L^2(\mathbb{R})$. By Lemma 2.2, we have

$$\begin{aligned}\langle f, \sigma(s,t)\psi_\ell\rangle &= W_{\psi_\ell} f(s,t)\\ &= \sqrt{s} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{\psi}(s\xi)} e^{2\pi i s t \xi}\, d\xi\\ &= \tfrac{1}{\sqrt{s}} \int_{-\Omega}^{\Omega} \hat{f}(\tfrac{\xi}{s})\overline{\hat{\psi}_\ell(\xi)} e^{2\pi i t \xi}\, d\xi.\end{aligned}$$

Then, using the fact that $\bigcup_{\ell=1}^{L} \mathcal{E}(T_\ell, v_\ell, r)$ constitutes a frame for $L^2[-\Omega,\Omega]$ with frame bounds say A' and B' and employing (5.15), we obtain

$$\begin{aligned}&\sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s) \sum_{t\in T_\ell} v_\ell(t)\, |\langle f, \sigma(s,t)\psi_\ell\rangle|^2\\ &= \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s) \sum_{t\in T_\ell} v_\ell(t) \left| \int_{-\Omega}^{\Omega} \left[\tfrac{1}{\sqrt{s}}\hat{f}(\tfrac{\xi}{s})\overline{\hat{\psi}_\ell(\xi)}\right] e^{2\pi i t\xi}\, d\xi \right|^2\\ &\le B' \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s) \int_{-\Omega}^{\Omega} \left|\tfrac{1}{\sqrt{s}}\hat{f}(\tfrac{\xi}{s})\hat{\psi}_\ell(\xi)\right|^2 d\xi\\ &= B' \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \sum_{\ell=1}^{L} \sum_{s\in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi)|^2 d\xi\\ &\le B'B \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi\\ &= B'B\, \|f\|_2^2 .\end{aligned}$$

This proves that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, S_\ell \times T_\ell, (w_\ell, v_\ell))$ is a Bessel sequence for $L^2(\mathbb{R})$. In a similar way we can also show that it possesses a lower frame bound.

Hence it suffices to prove that (ii) implies (5.15). First we show that $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$ implies the existence of some $B < \infty$ such that the second inequality in (5.15) is satisfied. For this, fix $\ell \in \{1, \ldots, L\}$. Since $\mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty$, in particular, $\mathcal{D}^+(S_\ell, w_\ell) < \infty$ by Proposition 4.17. Employing Lemma 5.10 shows that for some $\varepsilon > 0$ there exists $0 < \gamma < 1$ such that

$$\sum_{s\in S_\ell} w_\ell(s) |\hat{\psi}_\ell(s\xi)|^2 \chi_{[0,\gamma)}(s|\xi|) < \varepsilon \quad \text{for all } \xi \in \mathbb{R}.$$

We now focus on the sum $\sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2\chi_{[\gamma,\Omega)}(s|\xi|)$. Our hypotheses imply that there exists $M > 0$ satisfying $|\xi||\hat{\psi}_\ell(\xi)| \le M$ for each $\xi \in \mathbb{R}$. By Proposition 4.15, we have $\#_{w_\ell}(S_\ell \cap \xi[\gamma,\Omega)) \le N < \infty$ for all $\xi \in \mathbb{R}^+$. Hence, for each $\xi \in \mathbb{R}$, we get

$$\begin{aligned}\sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2\chi_{[\gamma,\Omega)}(s|\xi|) &\le \sum_{s\in S_\ell\cap|\xi|^{-1}[\gamma,\Omega)} w_\ell(s)\, M^2 s^{-2}|\xi|^{-2}\\ &\le M^2(|\xi|^{-1}\gamma)^{-2}N|\xi|^{-2}\\ &= M^2N\gamma^{-2}.\end{aligned}$$

This settles the claim.

Secondly, we employ the hypothesis $\mathcal{D}^-(\{(S_\ell,w_\ell)\}_{\ell=1}^L) > 0$. We claim that this implies that there exists $A > 0$ such that the first inequality in (5.15) is satisfied. First, by Proposition 4.16, there exists some interval $I \subseteq \mathbb{R}^+$ of positive finite measure and some positive constant N satisfying

$$\sum_{\ell=1}^L \#_{w_\ell}(S_\ell \cap \xi I) > N \quad \text{for all } \xi \in \mathbb{R}^+. \tag{5.16}$$

Since $\xi = 0$ is an isolated zero of $\hat{\psi}_\ell$ for all $\ell = 1,\ldots,L$, we can choose $\varepsilon > 0$ with $\hat{\psi}_\ell(\xi) \ne 0$ for each $\xi \in (0,\varepsilon)$, $\ell = 1,\ldots,L$. Let $\xi_0 \in \mathbb{R}^+$ be chosen so that $\xi_0 I \subseteq (0,\varepsilon)$. Since $\hat{\psi}_\ell$ is continuous, we have $|\hat{\psi}_\ell(\xi)| \ge \delta$ on $\xi_0 I$ for some $\delta > 0$ for all $\ell = 1,\ldots,L$. Now fix some $\xi \in \mathbb{R}^+$. Then (5.16) implies the existence of some $s_0 \in S_\ell$ for some $\ell \in \{1,\ldots,L\}$ such that $s_0 \in \xi^{-1}\xi_0 I$. This immediately yields

$$\sum_{\ell=1}^L \sum_{s\in S_\ell} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2 \ge \sum_{\ell=1}^L \sum_{s\in S_\ell\cap\xi^{-1}\xi_0 I} w_\ell(s)|\hat{\psi}_\ell(s\xi)|^2 \ge N\delta^2,$$

thereby proving our claim.

Now the implication (ii) $\Rightarrow$ (i) and the moreover-part follow from our considerations in the first part of this proof. □

Remark 5.12. We point out that a related result on sufficient conditions for irregular (weighted) wavelet frames was derived by Gröchenig in [62]. To emphasize the difference to our Theorem 5.11, we observe that the focus in [62, Thm. 1] is on the introduction of adaptive weights to compensate for local variations of the sequence of time-scale indices, thereby deriving a *weighted* wavelet frame, and does not employ the notion of density. The two results are distinct and complementary.

We briefly remark on whether it is possible to weaken the hypotheses of the previous proposition and on a possible improvement.

Remark 5.13. (a) If ξ is not an isolated zero of $\hat{\psi}$, it is easy to check that the implication (ii) $\Rightarrow$ (i) does not automatically hold. For instance, if we let $S = \{2^j\}_{j\in\mathbb{Z}}$, for which $\mathcal{D}^-(S) = \mathcal{D}^+(S) = \frac{1}{\ln 2}$, and define $\psi \in L^2(\mathbb{R})$ by $\hat{\psi} = \chi_{[1,\frac{3}{2})}$, then $\bigcup_{j\in\mathbb{Z}} 2^j[1, \frac{3}{2})$ does not cover $\mathbb{R}$. Hence $\mathcal{W}(\psi, S \times T)$ is not even complete for any $T \subseteq \mathbb{R}$.

(b) One might further ask, whether it is possible to include the values of the frame bounds of the frame from (i) in condition (ii). However, it is not too difficult to see that this is not possible. One reason is that in fact there exists an abundance of possibilities for choosing $\mathcal{E}(T, r)$ with different frame bounds as indicated by the moreover-part of Theorem 5.11, thus changing the frame bounds of $\mathcal{W}(\psi, S \times T, (w, 1))$ while S and w remain the same.

Now the question arises, whether it is also sufficient to consider density conditions concerning the existence of Parseval frames, in particular, whether for some sequences of translations and analyzing wavelets satisfying mild regularity conditions the associated wavelet frame is a Parseval frame provided the (weighted) sequence of scale indices possesses a positive finite uniform density. However, the following result shows that this would be too much to hope for. One reason for this is that Parseval frames are very sensitive to perturbations of the indices, but density is not (see Lemma 4.14).

Proposition 5.14. *For any $\psi \in L^1(\mathbb{R}) \cap L^2_A(\mathbb{R})$ with $|\hat{\psi}(\xi)| \le a|\xi|^\alpha$ as $|\xi| \to 0$ for some $a, \alpha > 0$, where $\xi = 0$ is an isolated zero of $\hat{\psi}$, and $|\hat{\psi}(\xi)| = 0$ for any $|\xi| \ge \Omega$, and for any $T \subseteq \mathbb{R}$ satisfying that $\mathcal{E}(T, r)$ is a frame for $L^2[-r, r]$, where $r > 2\Omega$, there exists $S \subseteq \mathbb{R}^+$ with positive finite uniform density such that $\mathcal{W}(\psi, S \times T)$ does not form a Parseval frame for $L^2(\mathbb{R})$.*

Proof. Let $\psi \in L^2(\mathbb{R})\backslash\{0\}$ and $T \subseteq \mathbb{R}$ be chosen such that they satisfy the hypotheses of the proposition. Further let Ω be chosen minimal with $|\hat{\psi}(\xi)| = 0$ for any $|\xi| \ge \Omega$. Let C and D denote the frame bounds of $\mathcal{E}(T, r)$ in $L^2[-r, r]$. By the hypotheses, $\hat{\psi}$ is continuous, hence there exists an interval $I \subseteq \mathbb{R}$ and $\delta > 0$ such that $|\hat{\psi}(\xi)| \ge \delta$ for all $\xi \in I$. Without loss of generality we can assume that $I \subseteq \mathbb{R}^+$ and that there exists $j_0 \ge 2$ so that I is a proper subset of $(\frac{\Omega}{2^{j_0-1}}, \frac{\Omega}{2^{j_0-2}}]$. Let $0 < \varepsilon < \frac{1}{D}$. Setting $m := \lceil (\frac{1}{C} - \frac{1}{D} + \varepsilon)/\delta^2 \rceil + 1$, we can choose m disjoint elements a_k, $1 \le k \le m$ such that there exists $U \subseteq (\frac{\Omega}{2^{j_0+1}}, \frac{\Omega}{2^{j_0}}]$ of positive measure satisfying that $a_k U \subseteq I$ for all $k = 1, \ldots, m$. In particular, this implies that $a_k > 2$ for $k = 1, \ldots, m$. Now define $S \subseteq \mathbb{R}^+$ by $S := \{2^j\}_{j\in\mathbb{Z}} \cup \{a_k\}_{k=1}^m$. An easy computation shows that S has a positive finite uniform density equal to $\frac{1}{\ln 2}$. By Proposition 5.3, it suffices to show that provided

$$\sum_{s\in S} |\hat{\psi}(s\xi)|^2 \ge \frac{1}{D} \quad \text{for all } \xi \in \mathbb{R}^+,$$

there exists a set $U \subseteq \mathbb{R}$ of positive measure with

$$\sum_{s\in S} |\hat{\psi}(s\xi)|^2 > \frac{1}{C} \quad \text{for all } \xi \in U.$$

Lemma 5.10 proves that there exists some $0 < \gamma < 1$ with

$$\sum_{s\in S} |\hat{\psi}(s\xi)|^2 \chi_{[0,\gamma)}(s\xi) < \varepsilon \quad \text{for all } \xi \in \mathbb{R}^+.$$

Noting that we can assume that $\gamma = 2^{-j_1}\Omega$ for some $j_1 \in \mathbb{Z}$ by choosing γ slightly smaller if necessary, we obtain that, for all $\xi \in (\frac{\Omega}{2}, \Omega]$,

$$\frac{1}{D} \le \sum_{s\in S} |\hat{\psi}(s\xi)|^2 = \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2 \le \sum_{j=-j_1}^{0} |\hat{\psi}(2^j\xi)|^2 + \varepsilon. \tag{5.17}$$

Now let $\xi \in U$. Using (5.17), we compute

$$\begin{aligned}\sum_{s\in S} |\hat{\psi}(s\xi)|^2 &= \sum_{j=-\infty}^{j_0} |\hat{\psi}(2^j\xi)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\xi)|^2 \\ &\ge \sum_{j=j_0-j_1}^{j_0} |\hat{\psi}(2^j\xi)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\xi)|^2 \\ &= \sum_{j=-j_1}^{0} |\hat{\psi}(2^j 2^{j_0}\xi)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\xi)|^2 \\ &\ge \frac{1}{D} - \varepsilon + m\delta^2 > \frac{1}{C}.\end{aligned}$$

Hence the proposition is proved. □

5.6 Existence of Special Weight Functions

In this section we examine whether there always exist weight functions such that Theorem 5.11(ii) is satisfied, i.e., whether for $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ we can always construct weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ such that

$$0 < \mathcal{D}^-(\{(S_\ell, w_\ell)\}_{\ell=1}^L) \le \mathcal{D}^+(\{(S_\ell, w_\ell)\}_{\ell=1}^L) < \infty.$$

For simplicity, we will only consider the case $L = 1$.

The following result shows that for any sequence S in $\mathbb{R}^+$, there always exists a weight function such that the associated upper weighted density is finite.

Proposition 5.15. *Let $S \subseteq \mathbb{R}^+$. Then there exists a weight function $w : S \to \mathbb{R}^+$ with*

$$\mathcal{D}^+(S, w) < \infty.$$

If $\mathcal{D}^-(S) > 0$, then there even exists a weight function $w : S \to \mathbb{R}^+$ such that

$$\mathcal{D}^-(S, w) = \mathcal{D}^+(S, w) = 1.$$

Proof. Let $S \subseteq \mathbb{R}^+$ be arbitrary and fix some $h > 0$. Then we define the weight function $w : S \to \mathbb{R}^+$ in the following way. For each $s \in S$, we set

$$w(s) = \begin{cases} \frac{h}{\#(S \cap e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}}))} & : \text{if } s \in e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}}), k \in \mathbb{Z}, \\ & \quad \text{and } \#(S \cap e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) \neq \emptyset, \\ 0 & : \text{else.} \end{cases}$$

Notice that w is constructed in such a way that either $\#_w(S \cap e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) = h$ or $\#_w(S \cap e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) = 0$ for any $k \in \mathbb{Z}$. Since $\{e^{kh}[e^{-\frac{h}{2}}, e^{\frac{h}{2}})\}_{k \in \mathbb{Z}}$ is a partition of $\mathbb{R}^+$, it is easy to see that $\mathcal{D}^+(S, w) \leq 1$.

Next assume that S satisfies $\mathcal{D}^-(S) > 0$. Then, by Proposition 4.16, there exists $h > 0$ such that $\inf_{x \in \mathbb{R}^+} \#(S \cap x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})) > 0$, i.e., each interval $x[e^{-\frac{h}{2}}, e^{\frac{h}{2}})$ contains at least one point from S. Defining w as above by choosing this particular h settles the claim. □

If $S \subseteq \mathbb{R}^+$ is chosen such that $\mathcal{D}^-(S) = 0$, we cannot construct a weight function to achieve a positive lower weighted density.

Proposition 5.16. *Let $S \subseteq \mathbb{R}^+$ with $\mathcal{D}^-(S) = 0$. Then for all weight functions $w : S \to \mathbb{R}^+$ we have $\mathcal{D}^-(S, w) = 0$.*

Proof. Fix an interval $I \subseteq \mathbb{R}^+$ with $0 < \mu_{\mathbb{R}^+}(I) < \infty$. By Proposition 4.16, we have

$$\inf_{x \in \mathbb{R}^+} \#(S \cap xI) = 0.$$

Hence there exists $x_0 \in \mathbb{R}^+$ with $S \cap x_0 I = \emptyset$. Let $w : S \to \mathbb{R}^+$ be arbitrary. Then we still have $\#_w(S \cap x_0 I) = 0$. Applying Proposition 4.16 again shows $\mathcal{D}^-(S, w) = 0$. □

5.7 Co-Affine Systems

In the following we will study wavelet systems which are constructed by interchanging the translation and dilation operator. Recall that we already studied a special case of co-affine systems arising from classical affine systems in Section 4.3. Here we consider more general co-affine systems. Let $S_1, \ldots, S_L \subseteq \mathbb{R}^+$ with weight functions $w_\ell : S_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Further, let $T_1, \ldots, T_L \subseteq \mathbb{R}$ and $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$. In this section we consider co-affine systems of the form

$$\begin{aligned}&\bigcup_{\ell=1}^{L}\{w_\ell(s)^{\frac12}T_tD_s\psi_\ell : (s,t)\in S_\ell\times T_\ell\}\\&=\bigcup_{\ell=1}^{L}\{w_\ell(s)^{\frac12}D_sT_{\frac{t}{s}}\psi_\ell : (s,t)\in S_\ell\times T_\ell\}\\&=\bigcup_{\ell=1}^{L}\mathcal{W}(\psi_\ell,\{(s,\tfrac{t}{s}) : (s,t)\in S_\ell\times T_\ell\},(w_\ell,1)).\end{aligned}$$

We first require the following lemma.

Lemma 5.17. *Let $T\subseteq\mathbb{R}$ be discrete. Then the following conditions are equivalent.*

(i) $T+\mathbb{Z}=T$.

(ii) *There exist pairwise disjoint $t_i\in[0,1)$, $i=1,\ldots,N$, with $T=\bigcup_{i=1}^{N}(t_i+\mathbb{Z})$.*

Proof. First assume that (i) holds. Then for each $t\in T$ we have $t+\mathbb{Z}\subseteq T$. Thus we obtain

$$T=\bigcup_{t\in T}(t+\mathbb{Z})=\bigcup_{t\in T}((t+\mathbb{Z})\cap[0,1))+\mathbb{Z}.$$

In order for T to be discrete, the set $(t+\mathbb{Z})\cap[0,1)$ must be finite. Hence there exist $t_i\in[0,1)$, $i=1,\ldots,N$, with $(t+\mathbb{Z})\cap[0,1)=\{t_i\}_{i=1}^{N}$.

Conversely, suppose that (ii) holds, and let $k\in\mathbb{Z}$. Then

$$T+k=\bigcup_{i=1}^{N}(t_i+\mathbb{Z})+k=\bigcup_{i=1}^{N}(t_i+k+\mathbb{Z})=T,$$

hence (i) holds. □

The following result on the non-existence of co-affine frames extends the result by Gressman, Labate, Weiss, and Wilson [61, Thm. 3] to weighted wavelet systems with finitely many generators, whose Fourier transforms satisfy a mild regularity condition, and with arbitrary sequences of scale indices.

Theorem 5.18. *Let $S_1,\ldots,S_L\subseteq\mathbb{R}^+$ with associated weight functions $w_\ell : S_\ell\to\mathbb{R}^+$ for $\ell=1,\ldots,L$ be given, and let $T_1,\ldots,T_L\subseteq\mathbb{R}$ be such that $T_\ell+\mathbb{Z}=T_\ell$ and $0\in T_\ell$ for $\ell=1,\ldots,L$. Further, let $\psi_1,\ldots,\psi_L\in L^1(\mathbb{R})\cap L^2_A(\mathbb{R})\backslash\{0\}$ with $|\hat{\psi}_\ell(\xi)|\le a|\xi|^\alpha$ as $|\xi|\to0$, and $|\hat{\psi}_\ell(\xi)|\le b|\xi|^{-\beta}$ as $|\xi|\to\infty$ for some $a,b,\alpha,\beta>0$ and for all $\ell=1,\ldots,L$. Then the weighted co-affine system*

$$\bigcup_{\ell=1}^{L}\mathcal{W}(\psi_\ell,\{(s,\tfrac{t}{s}) : (s,t)\in S_\ell\times T_\ell\},(w_\ell,1))$$

does not form a frame for $L^2(\mathbb{R})$.

Proof. Towards a contradiction assume that the system $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \{(s, \frac{t}{s}) : (s,t) \in S_\ell \times T_\ell\}, (w_\ell, 1))$ does constitute a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Fix some $f \in L^2(\mathbb{R})$ and define $F : \mathbb{R} \to \mathbb{R}^+$ by

$$F(x) = \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \sum_{t \in T_\ell} \left|\left\langle f, \sigma(s, \tfrac{t+x}{s})\psi_\ell \right\rangle\right|^2 .$$

Due to Lemma 5.17, for each $x \in \mathbb{R}$, we have

$$\begin{aligned} F(x+1) &= \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \sum_{t \in T_\ell} \left|\left\langle f, \sigma(s, \tfrac{t+x+1}{s})\psi_\ell \right\rangle\right|^2 \\ &= \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \sum_{i=1}^{N_\ell} \sum_{k \in \mathbb{Z}} \left|\left\langle f, \sigma(s, \tfrac{t_i+(k+1)+x}{s})\psi_\ell \right\rangle\right|^2 \\ &= F(x). \end{aligned}$$

Thus F is 1-periodic. Fix $\ell \in \{1, \dots, L\}$, $i \in \{1, \dots, N_\ell\}$, and $s \in S_\ell$. Using Lemma 2.2, we compute

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_0^1 \left|\left\langle f, \sigma(s, \tfrac{t_i+k+x}{s})\psi_\ell \right\rangle\right|^2 dx &= \sum_{k \in \mathbb{Z}} \int_{t_i}^{t_i+1} \left|\left\langle f, \sigma(s, \tfrac{k+x}{s})\psi_\ell \right\rangle\right|^2 dx \\ &= \int_{-\infty}^{\infty} \left|\left\langle f, \sigma(s, \tfrac{y}{s})\psi_\ell \right\rangle\right|^2 dy \\ &= \int_{-\infty}^{\infty} |W_{\psi_\ell} f(s, \tfrac{y}{s})|^2 dy \\ &= \int_{-\infty}^{\infty} |(\hat{f} \cdot D_{s^{-1}} \overline{\hat{\psi}_\ell})^\vee(y)|^2 dy \\ &= s \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |\hat{\psi}_\ell(s\xi)|^2 d\xi. \end{aligned}$$

Hence

$$\int_0^1 F(x)\, dx = \sum_{\ell=1}^{L} N_\ell \sum_{s \in S_\ell} w_\ell(s) s \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |\hat{\psi}_\ell(s\xi)|^2 d\xi.$$

Since $\|\sigma(1,-x)f\|_2 = \|f\|_2$ and

$$F(x) = \sum_{\ell=1}^{L} \sum_{s \in S_\ell} w_\ell(s) \sum_{t \in T_\ell} \left|\left\langle \sigma(1,-x)f, \sigma(s, \tfrac{t}{s})\psi_\ell \right\rangle\right|^2 ,$$

we have $A\|f\|_2^2 \le F(x) \le B\|f\|_2^2$ for all $x \in \mathbb{R}$. Together with the above computation and an appropriate choice for f we obtain

$$A \le \sum_{\ell=1}^{L} N_\ell \sum_{s\in S_\ell} w_\ell(s)s|\hat{\psi}_\ell(s\xi)|^2 \le B \quad \text{for all } \xi \in \mathbb{R}. \tag{5.18}$$

Now define $v_\ell : S_\ell \to \mathbb{R}^+$ by $v_\ell(s) = w_\ell(s)s$. Towards a contradiction, we assume that there exists $\ell_0 \in \{1, \ldots, L\}$ with $\mathcal{D}^+(S_{\ell_0}, v_{\ell_0}) = \infty$. Since $\hat{\psi}_{\ell_0}$ is continuous, there exists an interval $I \subseteq \mathbb{R}^+$ with positive finite measure such that $|\hat{\psi}_{\ell_0}(\xi)|^2 \ge \delta > 0$ for all $\xi \in I$. Applying Proposition 4.15, for each $n \in \mathbb{N}$, there exists some $\eta_n \in \mathbb{R}^+$ with $\#_{v_{\ell_0}}(S_{\ell_0} \cap \eta_n I) \ge n$. Hence,

$$\sum_{s\in S_{\ell_0}} v_{\ell_0}(s)|\hat{\psi}_{\ell_0}(s\eta_n^{-1})|^2 \ge \sum_{s\in S_{\ell_0}\cap\eta_n I} v_{\ell_0}(s)|\hat{\psi}_{\ell_0}(s\eta_n^{-1})|^2 \ge \delta\, n,$$

a contradiction to (5.18). Thus $\mathcal{D}^+(\{(S_\ell, v_\ell)\}_{\ell=1}^L) < \infty$ by Proposition 4.17. Therefore we can apply Proposition 5.4 to (5.18), which yields

$$\mathcal{D}^-(\{(S_\ell, C_{\psi_\ell}^+ N_\ell v_\ell)\}_{\ell=1}^L) \ge A > 0.$$

Thus, by Proposition 4.16, there exists an interval $I \subseteq \mathbb{R}^+$ with positive finite measure and $\delta > 0$ such that, for all $x \in \mathbb{R}^+$,

$$\sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \sum_{s\in S_\ell\cap xI} v_\ell(s) = \sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \,\#_{v_\ell}(S_\ell \cap xI) > \delta. \tag{5.19}$$

For brevity of notation we now define $\Lambda_\ell = \{(s, \frac{t}{s}) : (s,t) \in S_\ell \times T_\ell\}$ for all $\ell = 1, \ldots, L$. Since $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell, (w_\ell, 1))$ constitutes a frame for $L^2(\mathbb{R})$, by Theorem 4.1, it follows that $\mathcal{D}^+(\{(\Lambda_\ell, (w_\ell, 1))\}_{\ell=1}^L) < \infty$. By Proposition 3.4(i), this implies $\mathcal{D}^+(\Lambda_\ell, (w_\ell, 1)) < \infty$ for all $\ell = 1, \ldots, L$. Let $h > 0$ be such that $I \subseteq Q_h \cap (\mathbb{R}^+ \times \{0\})$. Now Proposition 3.6 implies that there exists $M < \infty$ with

$$\#_{(w_\ell,1)}(\Lambda_\ell \cap Q_h(x, 0)) < M \quad \text{for all } x \in \mathbb{R}^+ \text{ and } \ell = 1, \ldots, L.$$

Since for each $t \in T_\ell\backslash\{0\}$, $\frac{t}{s} \to \infty$ as $s \to 0$, and since $0 \in T_\ell$, there exists an $x_0 \in \mathbb{R}^+$ with $\Lambda_\ell \cap Q_h(x, 0) = S_\ell \times \{0\}$ for all $0 < x < x_0$ and $\ell = 1, \ldots, L$. Thus, in particular, there exists a constant C with

$$\sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \sum_{s\in S_\ell\cap xI} w_\ell(s) = \sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \,\#_{w_\ell}(S_\ell \cap xI) < CM \quad \text{for all } 0 < x < x_0.$$

Setting $I = [a, b]$, this implies

$$\begin{aligned}\sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \sum_{s\in S_\ell\cap xI} v_\ell(s) &= \sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \sum_{s\in S_\ell\cap xI} w_\ell(s)s \\ &\le xb \sum_{\ell=1}^{L} C_{\psi_\ell}^+ N_\ell \sum_{s\in S_\ell\cap xI} w_\ell(s) \\ &< xbCM \to 0 \quad \text{as } x \to 0,\end{aligned}$$

a contradiction to (5.19). □

We wish to mention that Proposition 4.10 does not follow as a corollary.

6

Homogeneous Approximation Property

The Homogeneous Approximation Property (HAP) is a key property of Gabor systems which not only leads to interesting approximation properties but also to necessary conditions for Gabor frames in terms of the Beurling density of the associated sequence of time-frequency indices. We show that, under some mild regularity assumptions, wavelet frames also satisfy an analogue of the HAP with respect to the affine group and that this leads to necessary conditions for existence in terms of the affine density. In so doing, essential differences from the Gabor case are also revealed: we see in the wavelet case how the density is strongly tied to the generator of the frame, and there is no Nyquist density.

We also obtain results on the HAP and related density conditions for wavelet systems that are Schauder bases but not Riesz bases.

The main results in this chapter generalize the results obtained in Heil and Kutyniok [75] to multiple generators.

6.1 Amalgam Spaces and the Continuous Wavelet Transform

For our purposes, we will need the following particular amalgam spaces on the affine group. For a brief review of amalgam spaces on locally compact groups, we refer to Section 2.4.

Our first goal is to derive an equivalent discrete-type norm for $W_{\mathbb{A}}(L^\infty, L^p)$. For this, we need the following notation. Given $h > 0$, define the following collection of translates of Q_h:

$$B_{jk} = B_{jk}(h) = Q_h(e^{jh}, khe^{-\frac{h}{2}}), \qquad j,k \in \mathbb{Z}. \tag{6.1}$$

Proposition 6.1. *If* $1 \le p < \infty$ *and* $h > 0$*, then the following is an equivalent norm for* $W_{\mathbb{A}}(L^\infty, L^p)$*:*

$$\|F\|_{W_{\mathbb{A}}(L^\infty,L^p)} = \Bigl(\sum_{j,k\in\mathbb{Z}} \|F \cdot \chi_{B_{jk}}\|_\infty^p\Bigr)^{\frac{1}{p}}. \tag{6.2}$$

Proof. Define $X = \{(e^{jh}, khe^{-\frac{h}{2}})\}_{j,k\in\mathbb{Z}}$. Lemma 3.5(i) says that X is Q_h-dense, and Lemma 3.5(ii) says that X is relatively separated. Lemma 3.5(iii) implies that there exists $N < \infty$ with

$$1 \le \sum_{m,n\in\mathbb{Z}} \chi_{Q_{2h}(e^{mh}, nhe^{-\frac{h}{2}})}(x,y) \le N \qquad \text{for all } (x,y) \in \mathbb{A}.$$

Consequently, if we set

$$\theta_{jk} = \frac{\phi((e^{jh}, khe^{-\frac{h}{2}})^{-1}\,\cdot)}{\sum_{m,n\in\mathbb{Z}} \phi((e^{mh}, nhe^{-\frac{h}{2}})^{-1}\,\cdot)},$$

where $\phi : \mathbb{A} \to \mathbb{R}$ is continuous with $0 \le \phi(x,y) \le 1$ for all $(x,y) \in \mathbb{A}$ satisfying that $\mathrm{supp}(\phi) \subseteq Q_{2h}$, and $\phi|_{Q_h} \equiv 1$, then $\{\theta_{jk}\}_{j,k\in\mathbb{Z}}$ is a BUPU. Therefore, Theorem 2.10 implies that the norm defined by $\|F\| = \bigl(\sum_{j,k} \|F \cdot \theta_{jk}\|_\infty^p\bigr)^{\frac{1}{p}}$ is an equivalent norm for $W_{\mathbb{A}}(L^\infty, L^p)$. Finally, because of Lemma 3.5(iii), this norm is equivalent to the desired norm (6.2). □

The amalgam space $W_{\mathbb{A}}(C, L^p)$ is the closed subspace of $W_{\mathbb{A}}(L^\infty, L^p)$ consisting of the continuous functions in $W_{\mathbb{A}}(L^\infty, L^p)$.

Corollary 6.2. *If $1 \le p \le q < \infty$, then $W_{\mathbb{A}}(L^\infty, L^p) \subseteq W_{\mathbb{A}}(L^\infty, L^q)$.*

We can now define the basic set of analyzing wavelets that we will use in the remainder of this chapter.

Definition 6.3. *The space $\mathcal{B}_0$ consists of all functions ψ on $\mathbb{R}$ which satisfy:*

(i) $|\psi(x)| \le C(1+|x|)^{-\alpha}$ *for some $C > 0$ and $\alpha > 2$,*
(ii) $\psi \in C^1(\mathbb{R})$, *i.e., ψ is differentiable, and ψ' is continuous and bounded, and*
(iii) $\hat{\psi}(0) = 0$.

The most important property of the class $\mathcal{B}_0$ is that its elements possess some time-scale concentration. This concentration is naturally measured by the amalgam space properties of the continuous wavelet transform, as given in the following theorem. The proof of this result is given in Section 6.2.

Theorem 6.4. (i) $\mathcal{B}_0 \subseteq L^2_A(\mathbb{R})$. *In particular, every element of $\mathcal{B}_0$ is admissible.*
(ii) *If f, $\psi \in \mathcal{B}_0$, then $W_\psi f \in W_{\mathbb{A}}(C, L^1)$.*
(iii) *If $\psi \in \mathcal{B}_0$, then $W_\psi \psi \in W_{\mathbb{A}}(C, L^1)$.*

While it is possible to construct wavelet frames for $L^2(\mathbb{R})$ using generators whose continuous wavelet transforms are not concentrated in time and scale, in practice such frames will have limited applicability. For example, in order that frame expansions converge in a range of function spaces rather than just L^2, or in order that the frame coefficients encode more properties of functions than just L^2-norm, requires analyzing wavelets with some regularity.

Note that if we let $\mathcal{S}_A = \{\psi \in \mathcal{S}(\mathbb{R}) : \hat{\psi}(0) = 0\}$ be the set of admissible Schwartz-class functions, then $\mathcal{S}_A \subseteq \mathcal{B}_0 \subseteq L^2_A(\mathbb{R})$. Thus the basic class $\mathcal{B}_0$ is dense in the set of admissible wavelets, which is itself dense in $L^2(\mathbb{R})$.

In Gabor analysis, the basic space of windows that can reasonably be used as generators of Gabor frames is the Feichtinger algebra S_0 [47], which is also known as the modulation space M^1 (we refer to [63] for details on the modulation spaces; compare also Section 2.3). The natural analog of S_0 for wavelet analysis would be the space $\mathcal{B}$ consisting of all functions ψ such that $W_\psi \psi \in W_{\mathbb{A}}(C, L^1)$. Our space $\mathcal{B}_0$ is slightly smaller, and we expect that our results should actually hold for all $\psi \in \mathcal{B}$, although we cannot yet prove this.

6.2 The Basic Class $\mathcal{B}_0$

In this section we will prove Theorem 6.4.

First, we require the following two results concerning decay of the CWT. The first result is similar to the result [41, Thm. 2.9.1] by Daubechies.

Theorem 6.5. *Assume that*

(i) $\int_{-\infty}^{\infty} (1 + |x|)\, |\psi(x)|\, dx < \infty$,

(ii) $f \in C^1(\mathbb{R})$, *i.e., f is differentiable and f' is continuous and bounded, and*

(iii) $\hat{\psi}(0) = 0$.

Then there exists $C > 0$ such that $|W_\psi f(a,b)| \le C\, a^{\frac{3}{2}}$ for all $(a,b) \in \mathbb{A}$.

Proof. By the Mean-Value Theorem, we have $|f(x) - f(y)| \le \|f'\|_\infty\, |x - y|$. Therefore,

$$\begin{aligned}
|W_\psi f(a,b)| &= \left| a^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x}{a} - b)}\, dx - a^{-\frac{1}{2}} f(ab) \int_{-\infty}^{\infty} \overline{\psi(\tfrac{x}{a} - b)}\, dx \right| \\
&\le a^{-\frac{1}{2}} \int_{-\infty}^{\infty} |f(x) - f(ab)|\, |\psi(\tfrac{x}{a} - b)|\, dx \\
&\le a^{-\frac{1}{2}}\, \|f'\|_\infty \int_{-\infty}^{\infty} |x - ab|\, |\psi(\tfrac{x-ab}{a})|\, dx \\
&= a^{-\frac{1}{2}}\, \|f'\|_\infty\, a^2 \int_{-\infty}^{\infty} |x\, \psi(x)|\, dx \\
&= C\, a^{\frac{3}{2}}.
\end{aligned}$$

This settles the claim. □

Theorem 6.6. *Assume that the functions ψ and f satisfy*

$$|\psi(x)| \le C(1+|x|)^{-\alpha} \qquad \textit{and} \qquad |f(x)| \le C(1+|x|)^{-\alpha}$$

for some $C > 0$ and $\alpha > 1$. Then there exists $C' > 0$ such that

$$|W_\psi f(a,b)| \le C' \, \frac{a^{\frac{1}{2}}}{1+a} \left(1 + \frac{a|b|}{1+a}\right)^{-\alpha} \qquad \textit{for all } (a,b) \in \mathbb{A}.$$

Proof. The wavelet transform $\widetilde{W}_\psi f$ used in Holschneider [83] is related to the wavelet transform of this book by the equality $\widetilde{W}_\psi f(a,b) = a^{-\frac{1}{2}} W_\psi f(a, \frac{b}{a})$. By [83, Thm. 11.0.2], we have

$$|\widetilde{W}_\psi f(a,b)| \le C' \, \frac{1}{1+a} \left(1 + \frac{|b|}{1+a}\right)^{-\alpha}.$$

A change of variables therefore completes the proof. □

We can now prove Theorem 6.4.

Proof (of Theorem 6.4). Assume that ψ, f belong to $\mathcal{B}_0$. In particular, we have that

(i) there exists $C > 0$ and $\alpha > 2$ such that $|f(x)|, |\psi(x)| \le C(1+|x|)^{-\alpha}$,
(ii) $f, \psi \in C^1(\mathbb{R})$, and
(iii) $\hat{f}(0) = \hat{\psi}(0) = 0$.

Since $\alpha > 2$, we have that $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Since we also have $\hat{\psi}(0) = 0$, this implies that ψ is admissible, cf. [41, p. 24].

Furthermore,

$$\int_{-\infty}^{\infty} (1+|x|)\,|\psi(x)|\,dx \le C \int_{-\infty}^{\infty} (1+|x|)^{1-\alpha}\,dx < \infty. \tag{6.3}$$

Consequently, Theorem 6.5 implies that there exists $C_1 > 0$ such that

$$|W_\psi f(a,b)| \le C_1 \, a^{\frac{3}{2}} \qquad \text{for all } (a,b) \in \mathbb{A}. \tag{6.4}$$

Additionally, by Theorem 6.6, there exists $C_2 > 0$ such that

$$|W_\psi f(a,b)| \le C_2 \, \frac{a^{\frac{1}{2}}}{1+a} \left(1 + \frac{a|b|}{1+a}\right)^{-\alpha} \qquad \text{for all } (a,b) \in \mathbb{A}. \tag{6.5}$$

Now set $h = 1$, and let $B_{jk} = B_{jk}(1) = Q_1(e^j, ke^{-\frac{1}{2}})$ as in (6.1). Since $\alpha > 2$, we can find γ such that $\frac{2\alpha-1}{\alpha-1} < \gamma < 3$. Set $N_j = e^{-\frac{\gamma j}{2}+1}$.

Define

$$S_1 = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty ,$$

$$S_2 = \sum_{j=-\infty}^{0} \sum_{|k|\le N_j} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty ,$$

$$S_3 = \sum_{j=-\infty}^{0} \sum_{|k|> N_j} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty .$$

We will show that $S_1, S_2, S_3 < \infty$. This implies by Proposition 6.1 that we have $W_\psi f \in W_{\mathbb{A}}(L^\infty, L^1)$, and since we already know that $W_\psi f$ is continuous, the proof will be complete.

Before doing this, however, let us make a generic observation. If we take a point $(a, b) \in B_{jk}$ for some j, $k \in \mathbb{Z}$, then

$$(a, b) = (e^j, ke^{-\frac{1}{2}})(x, y) = (e^j x, \tfrac{k}{x} e^{-\frac{1}{2}} + y)$$

for some $(x, y) \in Q_1 = [e^{-\frac{1}{2}}, e^{\frac{1}{2}}) \times [-\frac{1}{2}, \frac{1}{2})$. Therefore

$$e^{j-\frac{1}{2}} \le a \le e^{j+\frac{1}{2}} \qquad \text{and} \qquad \frac{|k|}{e} - \frac{1}{2} \le |b| \le |k| + \frac{1}{2}.$$

Estimate S_1. Suppose that $(a, b) \in B_{jk}$ with $j > 0$, $k \in \mathbb{Z}$. Then $a \ge 1$, so

$$1 + \frac{a|b|}{1+a} \ge 1 + \frac{|b|}{2} \ge 1 + \frac{|k|}{2e} - \frac{1}{4} = \frac{2|k| + 3e}{4e}.$$

Hence, we have from (6.5) that

$$|W_\psi f(a, b)| \le C_2 \, \frac{e^{\frac{j+\frac{1}{2}}{2}}}{1 + e^{j-\frac{1}{2}}} \left(\frac{4e}{2|k| + 3e}\right)^{\alpha} \le C_3 \, e^{-\frac{j}{2}} \, \frac{1}{(2|k| + 3e)^\alpha}.$$

Since $\alpha > 1$, we therefore have

$$S_1 = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty \le C_3 \sum_{j=1}^{\infty} e^{-\frac{j}{2}} \sum_{k\in\mathbb{Z}} \frac{1}{(2|k| + 3e)^\alpha} < \infty.$$

Estimate S_2. Suppose that $(a, b) \in B_{jk}$ with $j \le 0$, $|k| \le N_j = e^{-\frac{\gamma j}{2}+1}$. By (6.4), we have

$$|W_\psi f(a, b)| \le C_1 \, a^{\frac{3}{2}} \le C_4 \, e^{\frac{3}{2} j}.$$

Therefore, since $N_j \ge 1$ for all $j \le 0$, we have

$$S_2 = \sum_{j=-\infty}^{0} \sum_{|k|\le N_j} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty \le \sum_{j=-\infty}^{0} \sum_{|k|\le N_j} C_4\, e^{\frac{3}{2}j}$$

$$\le C_4 \sum_{j=-\infty}^{0} (2N_j+1) e^{\frac{3}{2}j}$$

$$\le 3eC_4 \sum_{j=-\infty}^{0} e^{-\frac{\gamma j}{2}+\frac{3}{2}j} < \infty,$$

the finiteness following from the fact that $\gamma < 3$.

Estimate S_3. If $(a,b) \in B_{jk}$ with $j \le 0$, $|k| > N_j = e^{-\frac{\gamma j}{2}+1}$, then, since $a \le 1$,

$$1 + \frac{a|b|}{1+a} \ge \frac{a|b|}{2} \ge \frac{e^{j-\frac{1}{2}}}{2}\left(\frac{|k|}{e} - \frac{1}{2}\right) = e^j \left(\frac{2|k|-e}{4e^{\frac{3}{2}}}\right).$$

Therefore, by (6.5) and the fact that $|k| > N_j \ge e$, we have

$$|W_\psi f(a,b)| \le C_2\, \frac{e^{\frac{j+\frac{1}{2}}{2}}}{1+0} \left(e^{-j}\, \frac{4e^{\frac{3}{2}}}{2|k|-e}\right)^\alpha = C_5\, e^{j(\frac{1}{2}-\alpha)}\, \frac{1}{(2|k|-e)^\alpha}.$$

Now, since $N_j \ge e$ we have for each $j \le 0$ that

$$\sum_{|k|>N_j} \frac{1}{(2|k|-e)^\alpha} \le 2\int_{N_j}^{\infty} \frac{1}{(2x-e)^\alpha}\, dx = \frac{1}{\alpha-1}\, \frac{1}{(2N_j-e)^{\alpha-1}}$$

$$\le \frac{1}{\alpha-1}\, N_j^{1-\alpha}$$

$$= \frac{e^{1-\alpha}}{\alpha-1}\, e^{\frac{\gamma j}{2}(\alpha-1)}.$$

Hence

$$S_3 = \sum_{j=-\infty}^{0} \sum_{|k|>N_j} \left\|W_\psi f \cdot \chi_{B_{jk}}\right\|_\infty \le \sum_{j=-\infty}^{0} \sum_{|k|>N_j} C_5\, e^{j(\frac{1}{2}-\alpha)}\, \frac{1}{(2|k|-e)^\alpha}$$

$$\le C_6 \sum_{j=-\infty}^{0} e^{j(\frac{1}{2}-\alpha)}\, e^{\frac{\gamma j}{2}(\alpha-1)}$$

$$= C_6 \sum_{j=-\infty}^{0} e^{\frac{j}{2}(1-2\alpha+\gamma(\alpha-1))}.$$

However,

$$1 - 2\alpha + \gamma(\alpha-1) > 1 - 2\alpha + \frac{2\alpha-1}{\alpha-1}\,(\alpha-1) = 0,$$

so we have $S_3 < \infty$. □

6.3 The Homogeneous Approximation Property for Wavelet Frames

In this section we define two versions of the HAP and prove that wavelet frames with generators from our basic class $\mathcal{B}_0$ satisfy the Strong HAP. Recall that if $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ is a frame for $L^2(\mathbb{R})$ then a canonical dual frame exists in $L^2(\mathbb{R})$, but that dual frame need not itself be a wavelet frame.

In the following we will denote the distance from $f \in L^2(\mathbb{R})$ to a closed subspace $V \subseteq L^2(\mathbb{R})$ by $\mathrm{dist}(f, V) = \inf\{\|f - v\| : v \in V\} = \|f - P_V f\|$, where P_V is the orthogonal projection of $L^2(\mathbb{R})$ onto V.

Definition 6.7. *Let $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ and $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ be such that the system $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell) = \{\sigma(a,b)\psi_\ell\}_{(a,b)\in\Lambda_\ell, \ell=1,\dots,L}$ is a wavelet frame for $L^2(\mathbb{R})$, and let its canonical dual frame be denoted by $\{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L}$. For each $h > 0$ and $(p,q) \in \mathbb{A}$, define a space*

$$W(h,p,q) = \mathrm{span}\big\{\tilde{\psi}_{a,b,\ell} : (a,b) \in Q_h(p,q) \cap \Lambda_\ell, \ell = 1, \dots, L\big\}. \tag{6.6}$$

(a) We say that the system $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ possesses the Weak Homogeneous Approximation Property (Weak HAP) *if for each $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} &\forall\, \varepsilon > 0, \quad \exists\, R = R(f,\varepsilon) > 0 \quad \textit{such that} \\ &\forall\, (p,q) \in \mathbb{A}, \quad \mathrm{dist}\Big(\sigma(p,q)f,\ W(R,p,q)\Big) < \varepsilon. \end{aligned} \tag{6.7}$$

(b) We say that the system $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ possesses the Strong Homogeneous Approximation Property (Strong HAP) *if given any $f \in L^2(\mathbb{R})$*

$$\begin{aligned} &\forall\, \varepsilon > 0, \quad \exists\, R = R(f,\varepsilon) > 0 \quad \textit{such that} \quad \forall\, (p,q) \in \mathbb{A}, \\ &\Big\|\sigma(p,q)f \ - \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \big\langle \sigma(p,q)f,\ \sigma(a,b)\psi_\ell \big\rangle\, \tilde{\psi}_{a,b,\ell} \Big\|_2 < \varepsilon. \end{aligned} \tag{6.8}$$

In either case we call $R(f,\varepsilon)$ an associated radius function.

Remark 6.8. (a) By Theorem 4.1, we have $\mathcal{D}^+(\bigcup_{\ell=1}^{L} \Lambda_\ell) < \infty$, and hence, by Proposition 3.4(i), it follows that $\mathcal{D}^+(\Lambda_\ell) < \infty$ for all $\ell = 1, \dots, L$. Hence there are only finitely many points of Λ_ℓ in each box $Q_h(p,q)$, and hence $W(h,p,q)$ is finite-dimensional.

(b) Since it is a linear combination of elements in $W(R,p,q)$, it is obvious that the function $\sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \langle \sigma(p,q)f,\ \sigma(a,b)\psi_\ell \rangle\, \tilde{\psi}_{a,b,\ell}$ is one element of the space $W(R,p,q)$, so the Strong HAP implies the Weak HAP.

(c) The terminology for Weak and Strong HAP used in the preceding definition is analogous to that used in Balan, Casazza, Heil, and Z. Landau [6], but differs from that used in Christensen, Deng, and Heil [22]. Specifically, the definition of "Strong HAP" in [22] was equivalent to the definition of "Weak HAP" in [22], and both of those are consistent with the definition of the Weak HAP used in this chapter.

First we require the following technical result.

Lemma 6.9. *Let $\delta > 0$ and $R' > 1$ be given, and define*

$$R = R'e^{\delta} + \delta e^{\delta} + \delta e^{2\delta}.$$

Then for every $(p,q) \in \mathbb{A}$ we have

$$Q_\delta(p,q) \setminus Q_R \neq \emptyset \implies Q_\delta(p,q) \cap Q_{R'} = \emptyset.$$

Proof. Suppose that $(p,q) \in \mathbb{A}$ and there exists $(a,b) \in Q_\delta(p,q) \setminus Q_R$. We must show that if $(c,d) \in Q_\delta$ then

$$(pc, \tfrac{q}{c} + d) = (p,q)(c,d) \notin Q_{R'}.$$

We proceed through cases, based on the facts that

$$(a,b) \notin Q_R = [e^{-\frac{R}{2}}, e^{\frac{R}{2}}) \times [-\tfrac{R}{2}, \tfrac{R}{2}), \tag{6.9}$$

$$(\tfrac{a}{p}, -\tfrac{pq}{a} + b) = (p,q)^{-1}(a,b) \in Q_\delta = [e^{-\frac{\delta}{2}}, e^{\frac{\delta}{2}}) \times [-\tfrac{\delta}{2}, \tfrac{\delta}{2}), \tag{6.10}$$

$$(c,d) \in Q_\delta = [e^{-\frac{\delta}{2}}, e^{\frac{\delta}{2}}) \times [-\tfrac{\delta}{2}, \tfrac{\delta}{2}). \tag{6.11}$$

Suppose that $a \geq e^{\frac{R}{2}}$. Then, using (6.9)–(6.11),

$$pc = \frac{p}{a}\, a\, c \geq e^{-\frac{\delta}{2}}\, e^{\frac{R}{2}}\, e^{-\frac{\delta}{2}} = e^{\frac{R}{2} - \delta} \geq e^{\frac{R'}{2}},$$

the last inequality following from the fact that $R = R'e^{\delta} + \delta e^{\delta} + \delta e^{2\delta} \geq R' + \delta + \delta$. Similarly, if $a < e^{-\frac{R}{2}}$ then $pc < e^{-\frac{R'}{2}}$. In either case we conclude that $(p,q)(c,d) \notin Q_{R'}$.

Now consider the case $b \geq \frac{R}{2}$. We have

$$\frac{q}{c} + d = -\frac{a}{p}\,\frac{1}{c}\Bigl(-\frac{pq}{a} + b\Bigr) + b\,\frac{a}{p}\,\frac{1}{c} + d \geq -e^{\frac{\delta}{2}} e^{\frac{\delta}{2}} \frac{\delta}{2} + \frac{R}{2} e^{-\frac{\delta}{2}} e^{-\frac{\delta}{2}} - \frac{\delta}{2} = \frac{R'}{2}.$$

Similarly, if $b < -\frac{R}{2}$ then $\frac{q}{c} + d < -\frac{R'}{2}$. In either case we conclude that $(p,q)(c,d) \notin Q_{R'}$. □

The following theorem now shows that provided the analyzing wavelets satisfy a mild regularity condition and the associated wavelet system forms a frame, then this system automatically fulfills the strong HAP.

Theorem 6.10. *Let $\psi_1, \dots, \psi_L \in \mathcal{B}_0$ and $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ be such that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ is a frame for $L^2(\mathbb{R})$. Then $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ satisfies the Strong HAP.*

Proof. Let A, B be frame bounds for $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$. In this case $\frac{1}{B}$, $\frac{1}{A}$ are frame bounds for the canonical dual frame $\{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L}$.

First we will show that the conditions of the Strong HAP, i.e., equation (6.8), are satisfied for functions in $\mathcal{B}_0$, and then extend by density to all of $L^2(\mathbb{R})$. Choose $g \in \mathcal{B}_0$ and fix $\varepsilon > 0$. Choose any $\delta > 0$. By Theorem 4.1, we have $\mathcal{D}^+(\bigcup_{\ell=1}^{L} \Lambda_\ell) < \infty$, and hence, by Proposition 3.4(i), $\mathcal{D}^+(\Lambda_\ell) < \infty$ for all $\ell = 1, \dots, L$. This implies that

$$M = \max_{\ell=1,\dots,L} \sup_{(x,y)\in\mathbb{A}} \#(\Lambda_\ell \cap Q_\delta(x,y)) < \infty.$$

Then for any $\ell \in \{1, \dots, L\}$ and $(p,q) \in \mathbb{A}$ we also have

$$\sup_{(x,y)\in\mathbb{A}} \#((p,q)^{-1}\cdot\Lambda_\ell \cap Q_\delta(x,y)) = \sup_{(x,y)\in\mathbb{A}} \#(\Lambda_\ell \cap Q_\delta((p,q)\cdot(x,y))) \le M < \infty.$$

Since g, $\psi_\ell \in \mathcal{B}_0$, it follows from Theorem 6.4 that $W_{\psi_\ell} g \in W_{\mathbb{A}}(C, L^1) \subseteq W_{\mathbb{A}}(C, L^2)$ for all $\ell = 1, \dots, L$. By Lemma 3.5, the sets $B_{jk} = B_{jk}(\delta)$ defined by (6.1) cover $\mathbb{A}$, with no element of this family intersecting more than $3(2e^\delta + 1)$ of the others. Considering the discrete-type norm for $W_{\mathbb{A}}(C, L^2)$ given in (6.2), we conclude that if R' is large enough and we set

$$J = \{(j,k) \in \mathbb{Z}^2 : B_{jk} \cap Q_{R'} = \emptyset\}, \tag{6.12}$$

then

$$\sum_{\ell=1}^{L} \sum_{(j,k)\in J} \|W_{\psi_\ell} g \cdot \chi_{B_{jk}}\|_\infty^2 < \frac{A\varepsilon^2}{M}. \tag{6.13}$$

Now set $R = R(g, \varepsilon) = R'e^\delta + \delta e^\delta + \delta e^{2\delta}$ and consider any point $(p,q) \in \mathbb{A}$. The function $\sigma(p,q)g$ has the frame expansion

$$\sigma(p,q)g = \sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda_\ell} \langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell \rangle\, \tilde{\psi}_{a,b,\ell}.$$

By applying equation (2.2) we have

$$\begin{aligned} &\Big\|\sigma(p,q)g \;- \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell \rangle\, \tilde{\psi}_{a,b,\ell}\Big\|_2^2 \\ &= \Big\|\sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda_\ell\setminus Q_R(p,q)} \langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell \rangle\, \tilde{\psi}_{a,b,\ell}\Big\|_2^2 \\ &\le \frac{1}{A} \sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda_\ell\setminus Q_R(p,q)} \big|\langle g,\, \sigma((p,q)^{-1}(a,b))\psi_\ell \rangle\big|^2 \end{aligned}$$

$$= \frac{1}{A} \sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda_\ell \setminus Q_R(p,q)} \left|W_{\psi_\ell} g((p,q)^{-1}(a,b))\right|^2$$

$$= \frac{1}{A} \sum_{\ell=1}^{L} \sum_{(c,d)\in(p,q)^{-1}\cdot\Lambda_\ell \setminus Q_R} \left|W_{\psi_\ell} g(c,d)\right|^2. \tag{6.14}$$

Now, each point $(c,d) \in (p,q)^{-1} \cdot \Lambda_\ell \setminus Q_R$ must lie in some set B_{jk}, and furthermore by Lemma 6.9 can only do so when $B_{jk} \cap Q_{R'} = \emptyset$, i.e., when $(j,k) \in J$. Moreover, each set B_{jk} can contain at most M elements of $(p,q)^{-1} \cdot \Lambda_\ell$. Hence, using (6.13), we can continue (6.14) as follows:

$$\frac{1}{A} \sum_{\ell=1}^{L} \sum_{(c,d)\in(p,q)^{-1}\Lambda_\ell \setminus Q_R} \left|W_{\psi_\ell} g(c,d)\right|^2 \le \frac{M}{A} \sum_{\ell=1}^{L} \sum_{(j,k)\in J} \left\|W_{\psi_\ell} g \cdot \chi_{B_{jk}}\right\|_\infty^2 < \varepsilon^2.$$

Thus (6.8) is satisfied for the function g.

Now suppose that f is any function in $L^2(\mathbb{R})$, and choose any $\varepsilon > 0$. Since $\mathcal{B}_0$ is dense in $L^2(\mathbb{R})$, there exists $g \in \mathcal{B}_0$ such that

$$\|f - g\|_2 < \frac{\varepsilon A^{\frac{1}{2}}}{3B^{\frac{1}{2}}}.$$

Set $R(f,\varepsilon) = R(g, \frac{\varepsilon}{3})$, and denote this quantity by R. Then for any $(p,q) \in \mathbb{A}$, we have

$$\begin{aligned}
&\Big\|\sigma(p,q)f - \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \langle \sigma(p,q)f,\, \sigma(a,b)\psi_\ell\rangle\, \tilde{\psi}_{a,b,\ell}\Big\|_2 \\
&\le \big\|\sigma(p,q)f - \sigma(p,q)g\big\|_2 \\
&\quad + \Big\|\sigma(p,q)g - \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell\rangle\, \tilde{\psi}_{a,b,\ell}\Big\|_2 \\
&\quad + \Big\|\sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \langle \sigma(p,q)g - \sigma(p,q)f,\, \sigma(a,b)\psi_\ell\rangle\, \tilde{\psi}_{a,b,\ell}\Big\|_2 \\
&< \frac{\varepsilon A^{\frac{1}{2}}}{3B^{\frac{1}{2}}} + \frac{\varepsilon}{3} + \left(\frac{1}{A} \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \left|\langle \sigma(p,q)g - \sigma(p,q)f,\, \sigma(a,b)\psi_\ell\rangle\right|^2\right)^{\frac{1}{2}} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left(\frac{B}{A} \left\|\sigma(p,q)g - \sigma(p,q)f\right\|_2^2\right)^{\frac{1}{2}} < \varepsilon.
\end{aligned}$$

In the above calculation, the second inequality uses the fact that g satisfies the Strong HAP and that $\{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L}$ has an upper frame bound of $\frac{1}{A}$. The third inequality follows from the fact that $\{\sigma(a,b)\psi_\ell\}_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L}$ has an upper frame bound of B. Thus (6.8) is satisfied for the function f, so $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ satisfies the Strong HAP. □

6.4 The Comparison Theorem for Wavelet Frames

We will see that wavelet frames satisfying the Weak HAP must fulfill certain density conditions with respect to other wavelet Riesz bases. In particular, by Theorem 6.10, these results apply to all wavelet frames $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ with generators $\psi_1, \dots, \psi_L \in \mathcal{B}_0$.

Note that in the following result, the reference Riesz basis $\mathcal{W}(\phi, \Delta)$ is not required to satisfy the HAP, so any Riesz basis can be used, including the classical affine orthonormal bases. However, there is a very important difference between this result and the analogous Comparison Theorem for Gabor systems [22, Thm. 3.6], namely that the density estimate depends on the value of the radius function associated to the frame $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$, whereas in the Gabor case it is independent of this value.

In our proof we will employ the double-projection technique from Ramanathan and Steger [107].

Theorem 6.11 (Comparison Theorem). *Assume that*

(i) $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ *and* $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ *are such that* $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ *is a frame for* $L^2(\mathbb{R})$ *that satisfies the* Weak HAP, *and*

(ii) $\phi \in L^2(\mathbb{R})$ *and* $\Delta \subseteq \mathbb{A}$ *are such that* $\mathcal{W}(\phi, \Delta)$ *is a Riesz basis for* $L^2(\mathbb{R})$.

Let $\{\tilde{\phi}_{a,b}\}_{(a,b)\in\Delta}$ *denote the canonical dual frame of* $\mathcal{W}(\phi, \Delta)$, *and set*

$$C = \sup_{(a,b)\in\Delta} \|\tilde{\phi}_{a,b}\|_2. \tag{6.15}$$

Then for each $\varepsilon > 0$, *by setting* $\Lambda = \bigcup_{\ell=1}^{L} \Lambda_\ell$, *we have*

$$\frac{1 - C\varepsilon}{e^{R(\phi,\varepsilon)}}\, \mathcal{D}^-(\Delta) \le \mathcal{D}^-(\Lambda) \quad \text{and} \quad \frac{1 - C\varepsilon}{e^{R(\phi,\varepsilon)}}\, \mathcal{D}^+(\Delta) \le \mathcal{D}^+(\Lambda).$$

Proof. Note that the elements of any frame are uniformly bounded in norm, so the value C defined in (6.15) is indeed finite. Let $\{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda_\ell, \ell=1,\dots,L}$ denote the canonical dual frame of $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$.

For each $h > 0$ and $(p,q) \in \mathbb{A}$, define

$$W(h,p,q) = \operatorname{span}\big\{\tilde{\psi}_{a,b,\ell} : (a,b) \in Q_h(p,q) \cap \Lambda_\ell, \ell = 1, \dots, L\big\},$$
$$V(h,p,q) = \operatorname{span}\big\{\sigma(a,b)\phi : (a,b) \in Q_h(p,q) \cap \Delta\big\}.$$

These spaces are finite-dimensional.

Fix any $\varepsilon > 0$, and let $R = R(\phi, \varepsilon)$ be the value such that (6.7) holds for the function $f = \phi$. Let $(p,q) \in \mathbb{A}$, $h > 0$, $\ell \in \{1, \dots, L\}$, and $(a,b) \in Q_h(p,q)$ be given. If $(x,y) \in Q_R(a,b) \cap \Lambda_\ell$, then since $Q_h \cdot Q_R \subseteq Q_{e^{\frac{R}{2}} h + R}$, we have

$$(x,y) \in Q_R(a,b) \subseteq (p,q) \cdot Q_h \cdot Q_R \subseteq Q_{e^{\frac{R}{2}} h + R}(p,q).$$

Thus $(x, y) \in Q_{e^{\frac{R}{2}}h+R}(p, q) \cap \Lambda_\ell$, which in turn implies

$$W(R, a, b) \subseteq W(e^{\frac{R}{2}}h + R, p, q).$$

Combining this with the definition of the Weak HAP, we see that

$$\operatorname{dist}\big(\sigma(a, b)\phi,\ W(e^{\frac{R}{2}}h + R, p, q)\big) \le \operatorname{dist}\big(\sigma(a, b)\phi,\ W(R, a, b)\big) < \varepsilon, \quad (6.16)$$

and this is valid for all $(p, q) \in \mathbb{A}$, $h > 0$, and $(a, b) \in Q_h(p, q)$.

Now let $h > 0$ and $(p, q) \in \mathbb{A}$ be fixed. Denote the orthogonal projections of $L^2(\mathbb{R})$ onto $V(h, p, q)$ and $W(e^{\frac{R}{2}}h + R, p, q)$ by P_V and P_W, respectively. Additionally, define the map $T : V(h, p, q) \to V(h, p, q)$ by $T = P_V P_W$. Since the domain of T is $V(h, p, q)$, we have $T = P_V P_W P_V$, and hence T is self-adjoint.

By definition, $\big\{\sigma(a, b)\phi : (a, b) \in Q_h(p, q) \cap \Delta\big\}$ is a basis for $V(h, p, q)$. Although the elements $\tilde{\phi}_{a,b}$ corresponding to the same indices need not lie in $V(h, p, q)$, their orthogonal projections are in that space, and we have for (a, b) and (c, d) in $Q_h(p, q) \cap \Delta$ that

$$\begin{aligned}
\big\langle \sigma(a, b)\phi,\ P_V(\tilde{\phi}_{c,d})\big\rangle &= \big\langle P_V(\sigma(a, b)\phi),\ \tilde{\phi}_{c,d}\big\rangle \\
&= \big\langle \sigma(a, b)\phi,\ \tilde{\phi}_{c,d}\big\rangle \\
&= \delta_{a,c}\,\delta_{b,d}. \qquad (6.17)
\end{aligned}$$

Since $V(h, p, q)$ is finite-dimensional, this implies that $\big\{P_V(\tilde{\phi}_{a,b}) : (a, b) \in Q_h(p, q) \cap \Delta\big\}$ is the dual basis to $\big\{\sigma(a, b)\phi : (a, b) \in Q_h(p, q) \cap \Delta\big\}$ in $V(h, p, q)$. Consequently, the trace of T is

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{(a,b)\in Q_h(p,q)\cap\Delta} \big\langle T(\sigma(a, b)\phi),\ P_V(\tilde{\phi}_{a,b})\big\rangle \\
&= \sum_{(a,b)\in Q_h(p,q)\cap\Delta} \big\langle P_V T(\sigma(a, b)\phi),\ \tilde{\phi}_{a,b}\big\rangle \\
&= \sum_{(a,b)\in Q_h(p,q)\cap\Delta} \big\langle T(\sigma(a, b)\phi),\ \tilde{\phi}_{a,b}\big\rangle.
\end{aligned}$$

Now, for $(a, b) \in Q_h(p, q) \cap \Delta$, we have

$$\begin{aligned}
&\big\langle T(\sigma(a, b)\phi),\ \tilde{\phi}_{a,b}\big\rangle \\
&= \big\langle P_V P_W(\sigma(a, b)\phi),\ \tilde{\phi}_{a,b}\big\rangle \\
&= \big\langle P_W(\sigma(a, b)\phi),\ P_V(\tilde{\phi}_{a,b})\big\rangle \\
&= \big\langle \sigma(a, b)\phi,\ P_V(\tilde{\phi}_{a,b})\big\rangle + \big\langle (P_W - I)(\sigma(a, b)\phi),\ P_V(\tilde{\phi}_{a,b})\big\rangle. \qquad (6.18)
\end{aligned}$$

By (6.17), the first term in (6.18) is

$$\big\langle \sigma(a,b)\phi,\, P_V(\tilde{\phi}_{a,b})\big\rangle = 1.$$

By the Cauchy–Schwarz inequality and equations (6.15) and (6.16), the second term in (6.18) is bounded by

$$\big|\big\langle (P_W - I)(\sigma(a,b)\phi),\, P_V(\tilde{\phi}_{a,b})\big\rangle\big| \le \|(P_W - I)(\sigma(a,b)\phi)\|_2 \; \|P_V(\tilde{\phi}_{a,b})\|_2 \le \varepsilon\, C.$$

This yields a lower bound for the trace of T:

$$\operatorname{tr}(T) \ge \sum_{(a,b)\in Q_h(p,q)\cap\Delta} (1 - C\varepsilon) = (1 - C\varepsilon)\,\#(Q_h(p,q)\cap\Delta).$$

On the other hand, the operator norm of T satisfies $\|T\| \le \|P_V\|\,\|P_W\| \le 1$, so all eigenvalues of T must satisfy $|\lambda| \le \|T\| \le 1$. This in turn provides us with an upper bound for the trace of T, because the trace is the sum of the nonzero eigenvalues, so

$$\operatorname{tr}(T) \le \operatorname{rank}(T) \le \dim\big(W(e^{\frac{R}{2}}h + R, p, q)\big) \le \#(Q_{e^{\frac{R}{2}}h+R}(p,q) \cap \textstyle\bigcup_{\ell=1}^{L} \Lambda_\ell).$$

Combining these two estimates, we see that for each $h > 0$ and all $(p,q) \in \mathbb{A}$, we have

$$(1 - C\varepsilon)\,\#(Q_h(p,q)\cap\Delta) \le \#(Q_{e^{\frac{R}{2}}h+R}(p,q)\cap\Lambda).$$

Therefore,

$$(1 - C\varepsilon)\,\frac{\#(Q_h(p,q)\cap\Delta)}{h^2} \le \frac{\#(Q_{e^{\frac{R}{2}}h+R}(p,q)\cap\Lambda)}{(e^{\frac{R}{2}}h+R)^2}\,\frac{(e^{\frac{R}{2}}h+R)^2}{h^2}.$$

Taking the infimum over all points $(p,q) \in \mathbb{A}$ and then the liminf as $h \to \infty$, or the supremum over all points $(p,q) \in \mathbb{A}$ and then the limsup as $h \to \infty$, therefore yields the estimates

$$(1 - C\varepsilon)D^-(\Delta) \le e^R\, D^-(\Lambda) \quad\text{and}\quad (1 - C\varepsilon)D^+(\Delta) \le e^R\, D^+(\Lambda).$$

Since we took $R = R(\phi, \varepsilon)$, this completes the proof. □

As a corollary, we obtain the following necessary density condition. This density condition was also obtained in Theorem 4.2, but with the restrictive additional hypothesis that $\mathcal{D}^+(\bigcup_{\ell=1}^{L} \Lambda_\ell^{-1}) < \infty$. Moreover, Theorem 6.11 provides more information, in terms of the value of the associated radius function, than merely the fact that $\mathcal{D}^-(\bigcup_{\ell=1}^{L} \Lambda_\ell)$ must be positive. In particular, the following result applies to any frame generated by analyzing wavelets $\psi_1, \dots, \psi_L \in \mathcal{B}_0$, since by Theorem 6.10, such a frame will satisfy the Strong (and hence the Weak) HAP. We further mention that a similar result for singly generated wavelet systems with analyzing wavelets satisfying a different regularity condition, however without any additional information on the value of the lower density, was derived by Sun and Zhou [119, Thm. 3.4] for their notion of density.

Corollary 6.12. *Let* $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ *and* $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ *be such that the system* $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell)$ *is a frame for* $L^2(\mathbb{R})$ *that satisfies the* Weak HAP*. Then, by setting* $\Lambda = \bigcup_{\ell=1}^L \Lambda_\ell$*, we have*

$$0 < \mathcal{D}^-(\Lambda) \le \mathcal{D}^+(\Lambda) < \infty.$$

Proof. The fact that $\mathcal{D}^+(\Lambda) < \infty$ follows from Theorem 4.1 and Remark 3.2(a). To show that the lower affine density is positive, let $\phi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)} \in L^2(\mathbb{R})$ be the Haar wavelet and set $\Delta = \{(2^j, k)\}_{j,k\in\mathbb{Z}}$. Then $\mathcal{W}(\phi, \Delta)$ is the classical Haar orthonormal basis for $L^2(\mathbb{R})$. Lemma 3.3 shows that $\mathcal{D}^-(\Delta) = \frac{1}{\ln 2}$. Therefore, Theorem 6.11 applied to the frame $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell)$ and the Haar basis $\mathcal{W}(\phi, \Delta)$ implies that for any $0 < \varepsilon < 1$ we have

$$\mathcal{D}^-(\Lambda) \ge \frac{1-\varepsilon}{e^{R(\phi,\varepsilon)} \ln 2} > 0,$$

which finishes the proof. □

6.5 Density Results for Wavelet Schauder Bases

In this section we will derive some results for wavelet systems which form Schauder bases for $L^2(\mathbb{R})$.

The existence of a wavelet Schauder basis (albeit requiring two generators) which is not a Riesz basis is shown in the following example.

Example 6.13. Fix $0 < \alpha < \frac{1}{2}$. Define a function ψ by $\hat{\psi}(\xi) = |\xi - \frac{3}{2}|^\alpha$ for $\xi \in [1, 2]$ and zero otherwise. It is a nontrivial result of Babenko [3] that $\{e^{2\pi i m\xi}\hat{\psi}(\xi)\}_{m\in\mathbb{Z}}$ forms a Schauder basis for $L^2[1, 2]$ (with respect to an appropriate ordering of $\mathbb{Z}$), but this Schauder basis is not a Riesz basis for $L^2([1, 2])$ (see also the discussion in Singer [114, pp. 351–354]). An argument similar to the one used in Deng and Heil [45, Ex. 3.3] then shows that if we set $\Lambda = \{(2^j, k)\}_{j,k\in\mathbb{Z}}$, then there exists an ordering of Λ such that $\mathcal{W}(\psi, \Lambda)$ is a Schauder basis but not a Riesz basis for $\mathcal{H}^2_+$, which is the space consisting of functions in $L^2(\mathbb{R})$ whose Fourier transforms are supported in $[0, \infty)$. If we define ψ_- by $\hat{\psi}_-(\xi) = \hat{\psi}(-\xi)$, then $\mathcal{W}(\psi, \Lambda) \cup \mathcal{W}(\psi_-, \Lambda)$ is a Schauder basis but not a Riesz basis for $L^2(\mathbb{R})$. However, it is possible to show that this system possesses an upper frame bound but not a lower frame bound, and hence its dual basis possesses a lower frame bound but not an upper frame bound.

We first show that the upper affine density of any wavelet Schauder basic sequence must be finite.

Proposition 6.14. *Let* $\psi_1, \dots, \psi_L \in L^2(\mathbb{R})$ *and* $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{A}$ *be such that the system* $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell, \Lambda_\ell)$ *is a Schauder basic sequence in* $L^2(\mathbb{R})$*. Then, by setting* $\Lambda = \bigcup_{\ell=1}^L \Lambda_\ell$*, we have* $\mathcal{D}^+(\Lambda) < \infty$*.*

Proof. Suppose that the sequence $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ is a Schauder basis for $Y = \overline{\text{span}}\{\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)\} \subseteq L^2(\mathbb{R})$. Schauder bases are countable, so let $\Lambda_\ell = \{(a_{\ell k}, b_{\ell k})\}_{k\in\mathbb{N}}$ be an ordering of Λ_ℓ for all $\ell = 1, \dots, L$ with respect to which the basis expansions converge in Y. Let $\tilde{\mathcal{W}} = \{\tilde{\psi}_{\ell k}\}_{\ell=1,\dots,L,k\in\mathbb{N}}$ denote the dual basis in Y. Let $S_{\ell N}(f) = \sum_{k=1}^{N} \langle f, \tilde{\psi}_{\ell k}\rangle\, \sigma(a_{\ell k}, b_{\ell k})\psi_\ell$ denote the associated partial sum operators for each $\ell = 1, \dots, L$.

Fix $\varepsilon > 0$. Since translation and dilation are strongly continuous families of operators on $L^2(\mathbb{R})$, there exists $\delta > 0$ such that

$$\|\sigma(a,b)\psi_\ell - \psi_\ell\|_2 = \|D_a T_b \psi_\ell - \psi_\ell\|_2 < \varepsilon \text{ for all } \ell = 1, \dots, L \text{ and } (a,b) \in Q_\delta.$$

Fix $\ell \in \{1, \dots, L\}$. Suppose that there exist two points $(a_{\ell m}, b_{\ell m})$ and $(a_{\ell n}, b_{\ell n})$ from Λ_ℓ that are both contained in some box $Q_\delta(x,y)$. Without loss of generality, assume $m < n$. If we define

$$\varphi_{m,n,\ell} = \sigma(a_{\ell m}, b_{\ell m})\psi_\ell - \sigma(a_{\ell n}, b_{\ell n})\psi_\ell,$$

then since $(x,y)^{-1} \cdot (a_{\ell m}, b_{\ell m}) \in Q_\delta$ and $(x,y)^{-1} \cdot (a_{\ell n}, b_{\ell n}) \in Q_\delta$, we have

$$\begin{aligned}
&\|\varphi_{m,n,\ell}\|_2 \\
&= \big\|\sigma((x,y)^{-1} \cdot (a_{\ell m}, b_{\ell m}))\psi_\ell - \sigma((x,y)^{-1} \cdot (a_{\ell n}, b_{\ell n}))\psi_\ell\big\|_2 \\
&\leq \big\|\sigma((x,y)^{-1} \cdot (a_{\ell m}, b_{\ell m}))\psi_\ell - \psi_\ell\big\|_2 + \big\|\sigma((x,y)^{-1} \cdot (a_{\ell n}, b_{\ell n}))\psi_\ell - \psi_\ell\big\|_2 \\
&< 2\varepsilon.
\end{aligned}$$

However, since $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ and $\tilde{\mathcal{W}}$ are biorthogonal,

$$\begin{aligned}
S_{\ell m}(\varphi_{m,n,\ell}) &= \sum_{k=1}^{m} \big\langle \sigma(a_{\ell m}, b_{\ell m})\psi_\ell,\, \tilde{\psi}_{\ell k}\big\rangle\, \sigma(a_{\ell k}, b_{\ell k})\psi_\ell \\
&\quad - \sum_{k=1}^{m} \big\langle \sigma(a_{\ell n}, b_{\ell n})\psi_\ell,\, \tilde{\psi}_{\ell k}\big\rangle\, \sigma(a_{\ell k}, b_{\ell k})\psi_\ell \\
&= \sigma(a_{\ell m}, b_{\ell m})\psi_\ell,
\end{aligned}$$

and therefore $\|S_{\ell m}(\varphi_{m,n,\ell})\|_2 = \|\sigma(a_{\ell m}, b_{\ell m})\psi_\ell\|_2 = \|\psi_\ell\|_2$. But then

$$\|S_{\ell m}\| = \sup_{\|f\|_2=1} \|S_{\ell m}(f)\|_2 \geq \frac{\|S_{\ell m}(\varphi_{m,n,\ell})\|_2}{\|\varphi_{m,n,\ell}\|_2} > \frac{\|\psi_\ell\|_2}{2\varepsilon}.$$

Since ε is arbitrary, this contradicts the fact that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ has a finite basis constant $C = \sup_{\ell,N} \|S_{\ell N}\|$.

Consequently, each translate $Q_\delta(x,y)$ can contain at most one point of Λ_ℓ for all $\ell = 1, \dots, L$. It follows from this that $D^+(\Lambda) < \infty$. □

The definition of the Strong and Weak HAPs for wavelet Schauder bases differs from the definition for frames only in that it uses the dual basis instead of the canonical dual frame. We use the same notation as before, except that $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell) = \{\sigma(a,b)\psi_\ell\}_{(a,b)\in\Lambda_\ell, \ell=1,\dots,L}$ is now assumed to be a Schauder basis, and $\{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L}$ is its dual basis. We define the space $W(h,p,q)$ by (6.6), using the dual basis instead of the canonical dual frame, and likewise make the corresponding minor changes in the definition of the Strong and Weak HAPs. By Proposition 6.14, we must have $\mathcal{D}^+(\bigcup_{\ell=1}^L \Lambda_\ell) < \infty$, and hence $W(h,p,q)$ will be finite-dimensional.

Note that since for each $\ell = 1,\dots,L$ all the elements of $\mathcal{W}(\psi_\ell,\Lambda_\ell)$ have exactly the same norm, the system $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$ is a bounded basis. Hence the dual basis is also a bounded basis, and thus the elements of the dual basis are uniformly bounded in norm.

We show now that if $\psi_1,\dots,\psi_L \in \mathcal{B}_0$ and $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$ is a Schauder basis for $L^2(\mathbb{R})$, then $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$ satisfies the Weak HAP.

Theorem 6.15. *Let $\psi_1,\dots,\psi_L \in \mathcal{B}_0$ and $\Lambda_1,\dots,\Lambda_L \subseteq \mathbb{A}$ be such that $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$ is a Schauder basis for $L^2(\mathbb{R})$. Then $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$ satisfies the* Weak HAP*.*

Proof. Let $\tilde{\mathcal{W}} = \{\tilde{\psi}_{a,b,\ell}\}_{(a,b)\in\Lambda,\ell=1,\dots,L}$ be the dual basis to $\bigcup_{\ell=1}^L \mathcal{W}(\psi_\ell,\Lambda_\ell)$. Then $C = \sup_{(a,b)\in\Lambda,\ell=1,\dots,L} \|\tilde{\psi}_{a,b,\ell}\|_2 < \infty$. Let

$$M = \max_{\ell=1\dots,L} \sup_{(x,y)\in\mathbb{A}} \#(\Lambda_\ell \cap Q_\delta(x,y)),$$

and note that $M < \infty$ by Propositions 6.14 and 3.4(i).

First we will show that the conditions of the Weak HAP, i.e., equation (6.7) are satisfied for functions in $\mathcal{B}_0$, then extend by density to all of $L^2(\mathbb{R})$. Choose $g \in \mathcal{B}_0$ and fix any $\varepsilon > 0$. Then, for any $\ell = 1,\dots,L$, we have $W_{\psi_\ell} g \in W_{\mathbb{A}}(C,L^1)$ by Theorem 6.4. Therefore, if we fix any $\delta > 0$, then we can find R' large enough that if J is defined by (6.12) then

$$\sum_{\ell=1}^{L} \sum_{(j,k)\in J} \left\| W_{\psi_\ell} g \cdot \chi_{B_{jk}} \right\|_\infty < \frac{\varepsilon}{CM}.$$

Now set $R = R(g,\varepsilon) = R'e^\delta + \delta e^\delta + \delta e^{2\delta}$. We then proceed similarly to the proof of Theorem 6.10, using the dual basis instead of the canonical dual frame and applying Minkowski's inequality instead of frame estimates. Choose any $(p,q) \in \mathbb{A}$. Since $Q_R(p,q) \cap \Lambda_\ell$ is a finite set for each $\ell = 1,\dots,L$, we have

$$\begin{aligned} \sigma(p,q)g \; &- \sum_{\ell=1}^{L} \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \left\langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell \right\rangle \tilde{\psi}_{a,b,\ell} \\ &= \sum_{\ell=1}^{L} \sum_{(a,b)\in\Lambda\setminus Q_R(p,q)} \left\langle \sigma(p,q)g,\, \sigma(a,b)\psi_\ell \right\rangle \tilde{\psi}_{a,b,\ell} \end{aligned}$$

with respect to some appropriate ordering of these series. Applying the triangle inequality, we therefore have

$$
\begin{aligned}
&\operatorname{dist}\bigl(\sigma(p,q)g,\ W(R,p,q)\bigr)\\
&\le \Bigl\|\sigma(p,q)g \ -\sum_{\ell=1}^{L}\ \sum_{(a,b)\in Q_R(p,q)\cap\Lambda_\ell} \bigl\langle \sigma(p,q)g,\ \sigma(a,b)\psi_\ell\bigr\rangle\, \tilde{\psi}_{a,b,\ell}\Bigr\|_2\\
&= \Bigl\|\sum_{\ell=1}^{L}\ \sum_{(a,b)\in\Lambda_\ell\setminus Q_R(p,q)} \bigl\langle \sigma(p,q)g,\ \sigma(a,b)\psi_\ell\bigr\rangle\, \tilde{\psi}_{a,b,\ell}\Bigr\|_2\\
&\le \Bigl(\sup_{(a,b)\in\Lambda_\ell,\ell=1,\dots,L} \|\tilde{\psi}_{a,b,\ell}\|_2\Bigr) \sum_{\ell=1}^{L}\ \sum_{(a,b)\in\Lambda_\ell\setminus Q_R(p,q)} \bigl|\bigl\langle g,\ \sigma((p,q)^{-1}(a,b))\psi_\ell\bigr\rangle\bigr|\\
&\le C\sum_{\ell=1}^{L}\ \sum_{(a,b)\in\Lambda_\ell\setminus Q_R(p,q)} \bigl|W_{\psi_\ell}g((p,q)^{-1}(a,b))\bigr|\\
&= C\sum_{\ell=1}^{L}\ \sum_{(c,d)\in(p,q)^{-1}\cdot\Lambda_\ell\setminus Q_R} \bigl|W_{\psi_\ell}g(c,d)\bigr|\\
&\le CM\sum_{\ell=1}^{L}\ \sum_{(j,k)\in J} \bigl\|W_{\psi_\ell}g\cdot\chi_{B_{jk}}\bigr\|_\infty < \varepsilon.
\end{aligned}
$$

Thus (6.7) is satisfied for the function g.

Now suppose that f is any function in $L^2(\mathbb{R})$, and choose any $\varepsilon > 0$. Since $\mathcal{B}_0$ is dense in $L^2(\mathbb{R})$, there exists $g \in \mathcal{B}_0$ such that $\|f-g\|_2 < \frac{\varepsilon}{3}$. Set $R(f,\varepsilon) = R(g,\frac{\varepsilon}{3})$, and denote this quantity by R. Choose any $(p,q)\in\mathbb{A}$, and let P_W denote the orthogonal projection of $L^2(\mathbb{R})$ onto $W(R,p,q)$. Then

$$
\begin{aligned}
&\operatorname{dist}\bigl(\sigma(p,q)f,\ W(R,p,q)\bigr)\\
&= \|\sigma(p,q)f - P_W\sigma(p,q)f\|_2\\
&\le \|\sigma(p,q)f - \sigma(p,q)g\|_2 + \|\sigma(p,q)g - P_W\sigma(p,q)g\|_2\\
&\quad + \|P_W\sigma(p,q)g - P_W\sigma(p,q)f\|_2\\
&\le \|f-g\|_2 + \operatorname{dist}\bigl(\sigma(p,q)g,\ W(R,p,q)\bigr) + \|g-f\|_2\\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus (6.7) is satisfied for the function f, so $\bigcup_{\ell=1}^{L}\mathcal{W}(\psi_\ell,\Lambda_\ell)$ possesses the Weak HAP. □

We can now compare the affine density of wavelet Schauder bases satisfying the Weak HAP with the affine density of other wavelet Schauder bases. The proof mostly follows the steps of the proof of Theorem 6.11, therefore we omit it here.

Theorem 6.16 (Comparison Theorem). *Assume that*

(i) $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ *and* $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ *are such that the system* $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ *is a Schauder basis for* $L^2(\mathbb{R})$ *that satisfies the* Weak HAP, *and*

(ii) $\phi \in L^2(\mathbb{R})$ *and* $\Delta \subseteq \mathbb{A}$ *are such that* $\mathcal{W}(\phi, \Delta)$ *is a Schauder basis for* $L^2(\mathbb{R})$.

Let $\{\tilde{\phi}_{a,b}\}_{(a,b)\in\Delta}$ *denote the dual basis of* $\mathcal{W}(\phi, \Delta)$. *Furthermore, set* $C = \sup_{(a,b)\in\Delta} \|\tilde{\phi}_{a,b}\|_2$. *Then for each* $\varepsilon > 0$, *by setting* $\Lambda = \bigcup_{\ell=1}^{L} \Lambda_\ell$, *we have*

$$\frac{1 - C\varepsilon}{e^{R(\phi,\varepsilon)}}\, \mathcal{D}^-(\Delta) \leq \mathcal{D}^-(\Lambda) \quad \textit{and} \quad \frac{1 - C\varepsilon}{e^{R(\phi,\varepsilon)}}\, \mathcal{D}^+(\Delta) \leq \mathcal{D}^+(\Lambda).$$

As a corollary, we obtain the following necessary density condition for the existence of wavelet Schauder bases that satisfy the Weak HAP.

Corollary 6.17. *Let* $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$ *and* $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ *be such that the system* $\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell)$ *is a Schauder basis for* $L^2(\mathbb{R})$ *that satisfies the* Weak HAP. *Then, by setting* $\Lambda = \bigcup_{\ell=1}^{L} \Lambda_\ell$, *we have*

$$0 < \mathcal{D}^-(\Lambda) \leq \mathcal{D}^+(\Lambda) < \infty.$$

Proof. In Theorem 6.16 we let $\phi \in L^2(\mathbb{R})$ be the Haar wavelet and $\Delta = \{(2^j, k)\}_{j,k\in\mathbb{Z}}$. Then $\mathcal{W}(\phi, \Delta)$ is an orthonormal basis for $L^2(\mathbb{R})$, and $\mathcal{D}^-(\Delta) = \mathcal{D}^+(\Delta) = \frac{1}{\ln 2}$. □

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